A Theory of Pfaffian Orientations I: Perfect Matchings and Permanents.

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Abstract

Kasteleyn stated that the generating function of the perfect matchings of a graph of genus g may be written as a linear combination of 4^g Pfaffians. Here we prove this statement. As a consequence we present a combinatorial way to compute permanents of square matrices.

Other consequences will be presented in a forthcoming paper.

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1 Introduction

We present a theory of Pfaffian orientations of graphs, introduced by Kasteleyn ([7, 6, 5]). Our approach is an extension of the treatment of toroidal rectangular lattices (see [4, 10, 7, 6, 5]). The case of general toroidal graphs was also studied by Barahona ([1]). As a consequence, we present a new technique to compute permanents of square matrices, which completes the Polya's scheme ([9]).

$G = (V, E)$ will always be a graph and x_e will be a variable associated with each edge e of G. We let $x = (x_e : e \in E)$ and for $M \subset E$ let $x(M)$ denote the product of the variables of the edges of M. An orientation of a graph G is a digraph obtained from G by fixing an orientation of each edge of G.

Let $A \Delta B$ denote the symmetric difference of the sets A and B and let $a \stackrel{2}{=} b$ denote $a = b$ modulo 2.

Definition 1.1 *The generating function of the perfect matchings of G is polynomial $\mathcal{P}(G, x)$ equal to the sum of $x(P)$ over all perfect matchings P of G.*

Definition 1.2 *Let G be a graph and let D be an orientation of G. Let M be a perfect matching of G. For each perfect matching P of G let $sgn(D, M\Delta P) = (-1)^n$ where n equals to the number of clockwise even alternating cycles of $M\Delta P$, and let $\mathcal{P}(D, M)$ equals to the sum of $sgn(D, M\Delta P)x(P)$ over all perfect matchings P of G.*

Definition 1.3 *Let $G = (V, E)$ be a graph with $2n$ vertices and D be an orientation of G. Denote by $A(D)$ the skew-symmetric matrix with the rows and the columns indexed by $|V|$, where $a_{vw} = x_{(v,w)}$ in case (v, w) is an arc of D, $a_{vw} = -x_{(v,w)}$ in case (w, v) is an arc of D, and $a_{vw} = 0$ otherwise.*

The Pfaffian of the skew-symmetric matrix $A(D)$ is defined as

$$Pf(A(D)) = \sum_P s^*(P) a_{i_1 j_1} \cdots a_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ is a partition of the set $\{1, \ldots, 2n\}$ into pairs, $i_k < j_k$ for $k = 1, \ldots, n$, and $s^*(P)$ equals to the sign of the permutation $(i_1 j_1 \ldots i_n j_n)$.

Each nonzero term of the expansion of the Pfaffian of $A(D)$ equals $x(P)$ or $-x(P)$ where P is a perfect matching of G. Let $s(D, P)$ equal the sign of the term $x(P)$ so that

$$Pf(A(D)) = \sum_P s(D, P) x(P)$$

The following theorem was proved by Kasteleyn ([5]).

Theorem 1.4 *Let G be a graph and D an orientation of G. Let P, M be two perfect matchings of G. Then*

$$s(D,P) = s(D,M)sgn(D, M\Delta P)$$

.

Hence,

$$Pf(A(D)) = \sum_P s(D,P)x(P) = s(D,M)\sum_P sgn(D, M\Delta P)x(P) = s(D,M)\mathcal{P}(D,M)$$

.

The following theorem is well-known (see [2]).

Theorem 1.5 *Let G be a graph and let D be an orientation of G.*
Then $Pf^2(A(D)) = det(A(D))$.

Remark. This theorem has an algorithmic application: if the edge-set of G is partitioned into a bounded number of classes and the variables x_e are equal in each class, then $\mathcal{P}(D,M)$ and $Pf(A(D))$ may be determined efficiently.

Let $M = \{\{i_1 j_1\}, \ldots, \{i_n j_n\}\}$, $i_k < j_k$, be a perfect matching of G. Let x' be defined as follows: $x'_e = x_e$, if $e \notin M$ and $x'_f = x_f z$, if $f \in M$, z is a new variable. Let A' be the matrix obtained from $A(D)$ by replacing each x_e by x'_e. Using Gaussian elimination we can express efficiently $det(A')$ as a rational function, i.e. a ratio of two polynomials. Since $det(A')$ is a polynomial, its coefficients can be determined efficiently from the rational function. We can view $det(A')$ as a polynomial $det(A')(x')$ or as a polynomial $det(A')(x,z)$. By Theorem 1.5, $Pf(A')(x,z) = \pm\sqrt{det(A')(x,z)}$. Hence we can determine efficiently a polynomial $Q(x,z)$ such that $Pf(A')(x,z) = \pm Q(x,z))$. Note that $\mathcal{P}(D,M) = \pm Q(x,1)$.

There is exactly one monomial in $Q(x,z)$ containing z^n and its coefficient is $+1$ or -1. Let $Q'(x,z)$ be the unique polynomial such that $Q'(x,z) = Q(x,z)$ or $Q'(x,z) = -Q(x,z)$ and the coefficient of $Q'(x,z)$ of the term containing z^n equals to $+1$. We have $\mathcal{P}(D,M) = +Q'(x,1)$. Moreover, $Pf(A(D)) = s(D,M)\mathcal{P}(D,M)$ and $s(D,M) = s^*(M)t^*(M)$ where $t^*(M)$ equals to the product of the signs of the elements $a_{i_k j_k}$ of the matrix $A(D)$ such that $i_k j_k \in M$. Hence $\mathcal{P}(D,M)$ and $Pf(A(D))$ may be determined efficiently.

Kasteleyn ([5]) introduced the following notion.

Definition 1.6 *A graph G is called Pfaffian if it has a Pfaffian orientation, i.e. an orientation such that each alternating cycle with respect to an arbitrary fixed perfect matching M of G is clockwise odd.*

Hence if a graph G has a Pfaffian orientation D then $s(D, P)$, P perfect matching of D, are equal and $\mathcal{P}(G, x)^2 = Pf^2(A(D)) = det(A(D))$.

Kasteleyn ([5]) also observed that the planar graphs have a Pfaffian orientation.

Theorem 1.7 *Each planar graph has a Pfaffian orientation.*

Proof. Let G be a planar graph, and let M be its perfect matching. Consider G drawn on the plane. Orient edges of G so that each face, except possibly the outer one, is clockwise odd. Each such face 'encircles' no vertex of G. Observe that the orientation has the property that a cycle C of G is clockwise odd if and only if C encircles an even number of vertices. Each alternating cycle with respect to M encircles an even number of vertices and hence it is clockwise odd. $\square$

2 The Perfect Matchings

Definition 2.1 *We define surface S_g , g positive integer, as follows. It consists of a base B_0 and $2g$ bridges B^i_j, $i = 1, ..., g$ and $j = 1, 2$. B_0 is a convex $4g$-gon with vertices $a_1, ..., a_n$, $n = 4g$, numbered clockwise. Bridge B^i_1 is a 4-gon with vertices $x^i_1, x^i_2, x^i_3, x^i_4$ numbered clockwise . It is glued with B_0 so that edge (x^i_1, x^i_2) of B^i_1 is identified with edge $(a_{4(i-1)+1}, a_{4(i-1)+2})$ of B_0 and edge (x^i_3, x^i_4) of B^i_1 is identified with edge $(a_{4(i-1)+3}, a_{4(i-1)+4})$ of B_0 .*

Bridge B^i_2 is a 4-gon with vertices $y^i_1, y^i_2, y^i_3, y^i_4$ numbered clockwise. It is glued with B_0 so that edge (y^i_1, y^i_2) of B^i_2 is identified with edge $(a_{4(i-1)+2}, a_{4(i-1)+3})$ of B_0 and edge (y^i_3, y^i_4) of B^i_2 is identified with edge $(a_{4(i-1)+4}, a_{4(i-1)+5 mod 4g})$ of B_0.

The definition of surface S_g corresponds to the usual definition of an orientable surface of genus g in the following sense. Orientable surface O_g of genus g may be obtained from S_g as follows: for each bridge B, glue together the two segments in which B intersects the boundary of B_0, and delete B.

Definition 2.2 *We say that a graph G is a g-graph if it may be drawn on S_g so that all the vertices belong to the base B_0, and each edge uses at most one bridge. The set of the edges drawn on the base will be denoted by $E_0 = E_0(G)$ and the set of edges drawn on bridge B^i_j will be denoted by $E^i_j = E^i_j(G)$.*

If moreover the following conditions are satisfied then we say that G is a proper g-graph.

1. *The outer face of the subgraph embedded on B_0 is a cycle, and it is embedded on the boundary of B_0,*

2. *Each vertex is incident with at most one edge out of E_0,*

3. *The subgraph embedded on B_0 has a perfect matching.*

If G is a proper g-graph then we denote by C_0 the cycle which forms the outer face of E_0, and we denote by M_0 a perfect matching of the subgraph of G embedded on B_0.

For each g-graph we fix its drawing on S_g.

Definition 2.3 *Let $G = (V, E)$ be a proper g-graph. The graphs $G_0 = (V, E_0)$ and $G^i_j = (V, E_0 \cup E^i_j)$ are planar. We define orientations of G_0 and G^i_j as follows: orientation D_0 of G_0 is Pfaffian and such that each face of G_0 is clockwise odd (as in the proof of Theorem 1.7). Next we draw G^i_j on the plane so that the drawing of G_0 is unchanged, and edge (x^i_1, x^i_4) ((y^i_1, y^i_4) respectively)of B^i_j belongs to the outer face of the drawing of G^i_j. Now complete D_0 to an orientation of G^i_j so that each face is clockwise odd. This defines orientation $+D^i_j$ of E^i_j.*

$-D^i_j$ is defined by reversing the orientation of D^i_j.

Remark 2.4 *If G is a proper g-graph and Pfaffian orientation D_0 is fixed, then D^i_j is uniquelly determined for each ij.*

Definition 2.5 *Let G be a proper g-graph, $g \geq 1$. An orientation D of G which equals to D^i_j or $-D^i_j$ on E^i_j and to D_0 on E_0 is called relevant. We define its type $r(D) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, \ldots, g-1$ and $j = 1, 2$, $r(D)_{2i+j}$ equals to $+1$ or -1 according to the sign of D^{i+1}_j in D.*

Definition 2.6 *Let $G = (V, E)$ be a proper g-graph and let A be a subset of its edges. We define its type $t(A) \in \{+1, -1\}^{2g}$ as follows. For $i = 0, .., g-1$ and $j = 1, 2$, we let $t(A)_{2i+j}$ equal to $(-1)^{s(A)_{2i+j}}$, where $s(A)_{2i+j}$ equals to the number of edges of A which belong to E^{i+1}_j.*

Let $CR(A) \stackrel{2}{=} \sum_{i=0}^{g-1} s(A)_{2i+1} \cdot s(A)_{2i+2}$.

Let $BR(A)$ be the subset of edges of A which do not belong to E_0.

For each $e \in BR(A)$, let $d(e) = 2i + j$ such that $e \in E^{i+1}_j$.

Let G be a proper g-graph. Any alternating cycle with respect to M_0 will be called *alternating cycle.* If C is a cycle of even length or a cycle embeddedd in the plane, then we denote by $l(C)$ the number of arcs of C directed clockwise, modulo 2.

We want to show that $sgn(D, M_0 \Delta P)$ depends only on $t(M_0 \Delta P)$ and $r(D)$.

Lemma 2.7 *Let G be a proper g-graph. Let $C_1, ..., C_k$ be vertex-disjoint cycles of G and let $\mathcal{C}$ denote their union. Then*

$$CR(\mathcal{C}) \stackrel{2}{=} \sum_{i=1}^{k} CR(C_i).$$

Proof. $CR(\mathcal{C})$ equals to the sum modulo 2 of $s(\mathcal{C})_{2i+1} \cdot s(\mathcal{C})_{2i+2}$, $i = 0, ..., g-1$. Now consider the drawing of cycles $C_1, ..., C_k$ in the plane, obtained by projecting each E^i_j outside of B_0. The total number of crossings modulo 2 also equals to the sum modulo 2 of $s(\mathcal{C})_{2i+1} \cdot s(\mathcal{C})_{2i+2}$, $i = 0, ..., g-1$. Each C_l, $l = 1, ..., k$ becomes a closed curve in the plane. Each pair of curves representing C_i and C_j intersect each other an even number of times. Hence we can forget these crossings without influencing the sum modulo 2 of $s(\mathcal{C})_{2i+1} \cdot s(\mathcal{C})_{2i+2}$, $i = 0, ..., g-1$. Each of the remaining crossings is a crossing of some C_l, $l = 1, ..., k$. □

Next we consider the case that there is exactly one alternating cycle of $M_0 \Delta P$.

Theorem 2.8 *Let G be a proper g-graph and let D be a relevant orientation of G. Let C be an alternating cycle of G. Then*

$$l(C) \stackrel{2}{=} [|BR(C)| - 1 - CR(C) + 1/2 \sum_{e \in BR(C)} (r(D)_{d(e)} + 1)].$$

Proof. We will assume without loss of generality that $G = C \cup C_0 \cup M_0$.

Claim 1. Let for each $i = 1, ...g$, C intersects at most one of E^i_1, E^i_2. Then

$$l(C) \stackrel{2}{=} [|BR(C)| - 1 + 1/2 \sum_{e \in BR(C)} (r(D)_{d(e)} + 1)].$$

Proof of Claim 1. C satisfying the properties of Claim 2 may be embedded in a planar way, by projecting the non-empty bridges outside of B_0. Hence $l(C) = 1$ iff $|\{e \in BR(C) : r(D)_{d(e)} = -1\}| \stackrel{2}{=} 0$. ¿From this Claim 1 follows.

End of Claim 1.

We procede by induction on $|BR(C)|$. The case $|BR(C)| = 0$ is considered in Claim 1.

Let theorem hold for all alternating cycles C' with $|BR(C)| > |BR(C')|$. Hence for such alternating cycles, $l(C')$ depends only on $t(C')$ and $r(D)$.

We let $l(t(F), r(H))$ denote the parity of the number of edges oriented clockwise of alternating cycle F with $|BR(F)| < |BR(C)|$, in relevant orientation H of a proper g-graph.

We will make the following notational agreement: if a segment S of C_0 is traversed clockwise then we denote it by $+S$, otherwise by $-S$.

If P is a path together with a prescribed way of traversing it, we denote by $l(P)$ the parity of the number of arcs of P oriented in agreement with the way of traversing P.

Claim 2. Let there be a bridge $B = B^i_j$ containing more than one edge of C. Then 2.8 holds.

Proof of Claim 2. Let e, f be two edges of C drawn on B which 'see' each other on B, i.e. there is no other edge of C drawn between them on B.

We remind that C_0 denotes the outer face of G_0 and e, f do not belong to $M_0 \subset E_0$.

Without loss of generality let e be nearer to edge $[a_{2(i-1)+j}, a_{2(i-1)+j+3}]$ of $B = B^i_j$ than f.

Let R be the cycle of G formed by e, f and two subpaths R_1, R_2 of C_0 defined by the endvertices of e, f. By the choice of e, f, R is a face of the planar drawing of $G^i_j = (V, E_0 \cup E^i_j)$. Observe that $l(R) = 1$ in the orientation of G^i_j induced by the relevant orientation D.

Let us consider new edge g (not belonging to G), between endpoints of e, f such that one of two cycles H_1, H_2 formed by g and C and containing g is alternating. Without loss of generality, let g use vertex $u \in R_1 \cap e$. Hence we have that $g = uv_1$ or $g = uv_2$ where $u \in R_1 \cap e$ and $v_i \in R_i \cap f$, $i = 1, 2$.

We denote two opposite orientations of g by g_1, g_2. Let H_1 use g_1 and H_2 use g_2.

Observe that $l(C) = l(H_1) + l(H_2)$.

We will assume without loss of generality that H_2 is alternating. Hence H_1 contains both e, f.

We distinguish two cases now.

Case 1: $g = uv_1$**.**

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B_0 along R_1.

Observe that $CR(C) \overset{2}{=} CR(H_1) + CR(H_2)$: let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are the same as the original edge-crossings of G; hence the parity of the number of edge-crossings of G' equals to $CR(C)$.

In addition to these crossings, H_1 and H_2 share two vertices of g_1, g_2. However, in our plane drawing these two vertices are touching points between the closed curves representing H_1, H_2, not crossing points. Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of H_1 and the number of crossings of H_2. The number of crossings of H_i modulo 2 equals $CR(H_i)$, $i = 1, 2$. Hence summarising $CR(C) \overset{2}{=} CR(H_1) + CR(H_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f, \}$ by adding new vertices u', v'_1 of degree 2, incident with new arcs e', f', g'_1. $e'f', g'_1$ are obtained from e, f, g_1 by replacing u by u' and v_1 by v'_1. We extend B_0 along R_2 and consider e', f', g'_1 embedded on extended B_0. Finally we add g_1' to M_0. Let H_1' be the cycle of D_1 obtained from H_1 by replacing e, f, g_1 by e', f', g_1'. We have $l(H_1) = l(H_1')$.

D_2 is obtained from $D - \{e, f,\}$ by adding arc g_2. We again extend B_0 along R_1 and consider g_2 embedded on extended B_0. We let $H_2' = H_2$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and H_i' is an alternating cycle of D_i. Moreover $|BR(H_i')| < |BR(C)|$. We also have that $CR(H_2) \stackrel{2}{=} CR(H_2')$ and $CR(H_1) \stackrel{2}{=} CR(H_1')$.

We remind that $l(R) = 1$ in the orientation of G_j^i induced by D. Hence, exactly one of g_i is oriented so that both cycles it makes with R are clockwise odd. Let it be g_2. Then D_2 is a relevant orientation and D_1 becomes relevant after reversing orientation of g_1. By the induction assumption for H_1', H_2', $l(H_2') = l(t(H_2'), r(D_2))$ and $l(H_1') \stackrel{2}{=} l(t(H_1'), r(D_1)) + 1$. We have that

$$l(C) \stackrel{2}{=} l(H_1) + l(H_2) \stackrel{2}{=} l(H_1') + l(H_2') \stackrel{2}{=}$$

$$l(t(H_1'), r(D_1)) + l(t(H_2'), r(D_2)) + 1 \stackrel{2}{=}$$

$$[|BR(C)| - 2 - 2 - CR(H_1') - CR(H_2') + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1)] + 1 \stackrel{2}{=}$$

$$[|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1)].$$

End of Case 1.

Case 2: $g = uv_2$.

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B.

Observe that $CR(C) + 1 \stackrel{2}{=} CR(H_1) + CR(H_2)$: let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are: the original crossings of G; the crossings of g_i and an edge $h \neq g_{2-i}$: but the total number of these crossings is even since each such edge h intersects both g_1, g_2.

Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, H_1 and H_2 share two vertices of g_1, g_2. In our plane drawing one of them is a crossing point and the other one a touching point between the closed curves representing H_1, H_2.

Hence, by the same argument as in the proof of Lemma 2.7, $CR(C) + 1$ equals the sum modulo 2 of the number of crossings of edges of H_1 and the number of crossings of the edges of H_2. The number of crossings of H_i modulo 2 equals to $CR(H_i)$, $i = 1, 2$. Hence summarising $CR(C) + 1 \stackrel{2}{=} CR(H_1) + CR(H_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D-\{e,f\}$ by adding new arc g_1' between v_1 and the endvertex u' of e different from u. If $l(fg_1e)=1$ then we let $g_1'=(v_1u')$. If $l(fg_1e)=0$ then $l(eg_1f)=1$ and we let $g_1'=(u',v_1)$.

We consider g_1' embedded on B. Let H_1' be obtained from H_1 by replacing f,g_1,e by g_1'. We have $l(H_1)=l(H_1')$.

D_2 is obtained from $D-\{e,f\}$ by adding arc g_2. We consider g_2 embedded on bridge B. We let $H_2=H_2'$.

Hence for $i=1,2$, D_i is orientation of a proper g-graph and H_i' is a cycle of D_i. Moreover $|BR(H_i')|<|BR(C)|$. We also have that $CR(H_2)=CR(H_2')$ and $CR(H_1)=CR(H_1')$.

We remind that $l(R)=1$ in the orientation of G_j^i induced by D. Hence exactly one of g_i is oriented so that both cycles it makes with R are clockwise odd. Let it be g_2. Hence D_2 is relevant with $r(D_2)=r(D)$ if and only if $r(D)_{2(i-1)+j}=1$, and we have:

$l(H_2)=l(t(H_2'),r(D))$ if and only if $r(D)_{2(i-1)+j}=1$. Otherwise $l(H_2)\overset{2}{=}l(t(H_2'),r(D))+1$.

We will show that the same holds for H_1': let R_3 be the segment of C_0 such that $H=(e,-R_1,-R_3,-R_2)$ is a cycle.

We will show that $l(g_1',-R_3,-R_2)=1$: $l(g_1',-R_3,-R_2)=l(e,g_1,f,-R_3,-R_2)\overset{2}{=}l(f,-R_3)+l(e,g_1,-R_2)$.

If $r(D)_{2(i-1)+j}=1$ then $l(f,-R_3)=1$ and $l(e,g_1,-R_2)=0$ since $l(e,g_2,-R_2)=1$.

If $r(D)_{2(i-1)+j}=-1$ then $l(f,-R_3)=0$ and $l(e,g_1,-R_2)=1$ since $l(e,g_2,-R_2)=0$.

Summarising $l(g_1',-R_3,-R_2)=1$.

Hence D_1 is relevant with $r(D_1)=r(D)$ if and only if $r(D)_{2(i-1)+j}=1$, and we again have:

$l(H_1)=l(H_1')=l(t(H_1'),r(D))$ if and only if $r(D)_{2(i-1)+j}=1$. Otherwise $l(H_1)=l(H_1')\overset{2}{=}l(t(H_1'),r(D))+1$.

Summarising and using the induction assumption of 2.8 for H_1',H_2' we get:

$$l(C)\overset{2}{=}l(H_1)+l(H_2)\overset{2}{=}$$
$$l(t(H_1'),r(D))+l(t(H_2'),r(D))\overset{2}{=}$$
$$[|BR(C)|-2-CR(C)-1+$$
$$1/2\sum_{h\in BR(C)-\{e,f\}}(r(D)_{d(h)}+1)+$$
$$1/2(r(D)_{d(g_2)}+1)+$$
$$1/2(r(D)_{d(g_1')}+1)]\overset{2}{=}$$
$$[|BR(C)|-1-CR(C)+1/2\sum_{h\in B(C)}(r(D)_{d(h)}+1)].$$

End of Case 2.
End of Claim 2.

Claim 3. Let there be i such that C contains exactly one edge from both E_1^i and E_2^i. Then theorem 2.8 holds.

Proof of Claim 3. Let e, f be the two edges on bridges E_1^i and E_2^i, respectively, and let C_1 and C_2 be two paths such that $C = (C_1, e, C_2, f)$.

The end-vertices of e, f belong to C_0. Let us assume that along the boundary of B_0 from $a_{4(i-1)+1}$ to a_{4i+1}, the endvertices of e, f appear in the order (v_1, u_1, v_2, u_2) where $e = u_1u_2$ and $f = v_1v_2$.

Let R_1, R_2 be two disjoint subpaths of the segment of C_0 between $a_{4(i-1)+1}$ and a_{4i+1}, which cover the endvertices of e, f. R_1, R_2 contain no other vertex of G incident with an edge out of E_0, by the choice of i.

R_1, R_2, e, f form a cycle R, and without loss of generality, let $R = (+R_1, e, -R_2, f)$ where R_1 is traversed clockwise, i.e. in agreement with the indices of the vertices along C_0, and R_2 is traversed anticlockwise.

Let us consider new adge g (not belonging to G), between endpoints of e, f such that one of two cycles I_1, I_2 formed by g and C and containing g is alternating. Without loss of generality let g use vertex $u_1 \in R_1 \cap e$. Hence we have that $g = u_1v_1$ or $g = u_1v_2$ where $u_1 \in R_1 \cap e$ and $v_i \in R_i \cap f$, $i = 1, 2$.

We denote two opposite orientations of g by g_1, g_2. Let I_1 use g_1 and I_2 use g_2.

Observe that $l(C) \stackrel{2}{=} l(I_1) + l(I_2)$.

We will assume without loss of generality that I_2 is alternating. Hence I_1 contains both e, f.

Let R_3 denote the segment of C_0 between u_1 and v_2.

Again we distinguish two cases.

Case 1: $g = u_1v_1$**.**

In this case g forms a cycle with R_1.

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B_0 along R_1.

Observe that $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$: let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are the same as the original edge-crossings of G. Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, I_1 and I_2 share two vertices of g_1, g_2. However, in our plane drawing these two vertices are touching points between the closed curves representing I_1, I_2, not crossing points. Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of I_1 and the number of crossings of I_2. The number of crossings of I_i equals to $CR(I_i)$, $i = 1, 2$. Hence summarising $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f\}$ by adding new vertices u_1', v_1' of degree 2, incident with new arcs e', f', g_1'. $e'f', g_1'$ are obtained from e, f, g_1 by replacing u_1 by u_1' and v_1 by v_1'. We extend B_0 along R_2 and consider e', f', g_1' embedded on extended B_0. Finally we add g_1' to M_0. Let I_1' be the cycle of D_1 obtained from I_1 by replacing e, f, g_1 by e', f', g_1'. We have $l(I_1) = l(I_1')$.

D_2 is obtained from $D - \{e, f\}$ by adding arc g_2. We again extend B_0 along R_1 and consider g_2 embedded on extended B_0. We let $I_2' = I_2$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and I_i' is an alternating cycle of D_i. Moreover $|BR(I_i')| < |BR(C)|$. We also have that $CR(I_2) = CR(I_2')$ and $CR(I_1) - 1 \overset{2}{=} CR(I_1')$.

Let us assume without loss of generality that g_2 is directed so that the cycle $l(-R_1, g_2) = 1$. Hence D_2 is a relevant orientation.

Observe that D_1 is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$:

first we prove if $r(D)_{d(e)} = r(D)_{d(f)} = 1$ then D_1 is a relevant orientation.

In this case we need to show that $l(-R_2, f', g_1', e') = 1$.

We have $l(g_1', f', -R_3) = l(g_1, f, -R_3) \overset{2}{=} l(g_2, f, -R_3) + 1 = 0$ since $r(D_{d(f)}) = 1$ and thus $l(-R_3, g_2, f) = 1$. Moreover $l(g_1', f') = l(f', g_1')$ since it does not matter which way two consecutive arcs are traversed. Hence $l(f', g_1') = l(-R_3)$.

Moreover $l(-R_2, -R_3, e') = l(-R_2, -R_3, e) = 1$ since $r(D)_{d(e)} = 1$. Replacing $f'g_1'$ for $-R_3$ gives what we needed.

Analogously, if $r(D)_{d(e)} = r(D)_{d(f)} = -1$ then $l(-R_2, f', g_1', e') = 1$ and D_1 is a relevant orientation.

On the other hand, if $r(D)_{d(e)} \neq r(D)_{d(f)}$ then D_1 is obtained from a relevant orientation by reversing one arc; hence D_1 is not relevant.

Summarizing,

$$l(I_1) + l(I_2) \overset{2}{=} l(I_1') + l(I_2') \overset{2}{=} l(t(I_1'), r(D_1)) + l(t(I_2'), r(D_2)) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1).$$

Using the induction assumption of 2.8 for I_1', I_2' we get:

$$l(C) \overset{2}{=} l(I_1) + l(I_2) \overset{2}{=}$$

$$[l(t(I_1'), r(D_1)) + l(t(I_2'), r(D_2)) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1)] \overset{2}{=}$$

$$[|BR(C)| - 4 - CR(C) + 1 + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1)] \overset{2}{=}$$

$$[|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1)].$$

End of Case 1.

Case 2: $g = uv_2$.

In this case g forms a cycle with R_3.

Let $G' = G \cup \{g_1, g_2\}$. We consider g_1, g_2 embedded on B_0 along R_3.

Observe that $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$: Let us project all the edges of $G' - E_0$ outside of B_0. The edge-crossings are the same as the original crossings of G.

Hence the parity of the number of edge-crossings equals $CR(C)$.

In addition to these crossings, I_1 and I_2 share two vertices of g_1, g_2. In the plane drawing both of them are touching points between the closed curves representing I_1, I_2.

Hence, by the same argument as in the proof of Lemma 2.7, $CR(C)$ equals the sum modulo 2 of the number of crossings of I_1 and the number of crossings of I_2. The number of crossings of I_i equals to $CR(I_i)$, $i = 1, 2$. Hence summarising $CR(C) \stackrel{2}{=} CR(I_1) + CR(I_2)$.

We construct two digraphs D_1, D_2 as follows:

D_1 is obtained from $D - \{e, f\}$ by adding new vertices u_1', v_2' of degree 2, incident with new arcs e', f', g_1'. e', f', g_1' are obtained from e, f, g_1 by replacing u_1 by u_1' and v_2 by v_2'. We extend B_0 along $+R_1 + R_3 + R_2$ and consider e', f', g_1' embedded on extended B_0. Finally we add g_1' to M_0. Let I_1' be the cycle of D_1 obtained from I_1 by replacing e, f, g_1 by e', f', g_1'. We have $l(I_1) = l(I_1')$.

D_2 is obtained from $D - \{e, f, \}$ by adding arc g_2. We again extend B_0 along R_3 and consider g_2 embedded on extended B_0. We let $I_2' = I_2$.

Hence for $i = 1, 2$, D_i is orientation of a proper g-graph and I_i' is an alternating cycle of D_i. Moreover $|BR(I_i')| < |BR(C)|$. We also have that $CR(I_2) = CR(I_2')$ and $CR(I_1) - 1 \stackrel{2}{=} CR(I_1')$.

Let us assume without loss of generality that g_2 is directed so that $l(-R_3, g_2) = 1$. Hence D_2 is a relevant orientation.

Observe that D_1 is a relevant orientation if and only if $r(D)_{d(e)} = r(D)_{d(f)}$:

It again suffices to consider the case that $r(D)_{d(e)} = r(D)_{d(f)} = +1$.

In this case we need to show that $l(-R_2, -R_3, -R_1, f', g_1', e') = 1$:

We have $l(-R_1, f', g_1') = 0$ since $r(D_{d(f)}) = 1$ and thus $l(-R_1, f, g_2) = 1$.

Moreover $l(-R_2, -R_3, e') = l(-R_2, -R_3, e) = 1$ since $r(D)_{d(e)} = +1$.

Summarising $l(-R_2, -R_3, -R_1, f', g_1', e') = 1$.

Hence

$$l(I_1) + l(I_2) \stackrel{2}{=} l(I_1') + l(I_2') \stackrel{2}{=} l(t(I_1'), r(D_1)) + l(t(I_2'), r(D_2)) +$$

$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1).$$

Using the induction assumption of 2.8 for I_1, I_2 we get:

$$l(C) \stackrel{2}{=} l(I_1) + l(I_2) \stackrel{2}{=}$$
$$l(t(I_1'), r(D_1)) + l(t(I_2'), r(D_2)) +$$
$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \stackrel{2}{=}$$
$$|BR(C)| - 4 - CR(C) + 1 + 1/2 \sum_{h \in BR(C) - \{e,f\}} (r(D)_{d(h)} + 1) +$$
$$1/2(r(D)_{d(e)} - 1 + r(D)_{d(f)} - 1) \stackrel{2}{=}$$
$$|BR(C)| - 1 - CR(C) + 1/2 \sum_{h \in BR(C)} (r(D)_{d(h)} + 1).$$

End of Case 2.

End of Claim 3.

Theorem follows from Claim 1, 2, 3. □

Next we show that the statement of Theorem 2.8 holds for the set of the alternating cycles of $M_0 \Delta P$ as well.

Theorem 2.9 *Let G be a proper g-graph and let D be a relevant orientation of G. Let P be a perfect matching of G. Then*

$$sgn(D, M_0 \Delta P) = (-1)^q,$$

where

$$q \stackrel{2}{=} [|BR(M_0 \Delta P)| - CR(M_0 \Delta P) +$$
$$1/2 \sum_{e \in BR(M \Delta P)} (r(D)_{d(e)} + 1)].$$

Proof. Let $C_1, ..., C_k$ be the alternating cycles of $M_0 \Delta P$. We have $sgn(D, M_0 \Delta P) = (-1)^q$, where $q \stackrel{2}{=} l(C_1) + ... + l(C_k) - k$.

Using Theorem 2.8 for $C_1, ..., C_k$ it remains to show:

$$CR(M_0 \Delta P) \stackrel{2}{=} \sum_{j=1}^{k} CR(C_j).$$

This is true by Lemma 2.7.

□

Corollary 2.10 *Let G be a proper 1-graph and D a relevant orientation of G. Let C be an alternating cycle with respect to M_0. Then*

1. *if $r(D) = (1,1)$ then $sgn(D,C) = 1$ iff $t(C) = (1,1)$ or $t(C) = (1,-1)$ or $t(C) = (-1,1)$.*
2. *If $r(D) = (1,-1)$ then $sgn(D,C) = 1$ iff $t(C) = (1,1)$ or $t(C) = (-1,-1)$ or $t(C) = (-1,1)$.*
3. *If $r(D) = (-1,1)$ then $sgn(D,C) = 1$ iff $t(C) = (1,1)$ or $t(C) = (-1,-1)$ or $t(C) = (1,-1)$.*
4. *If $r(D) = (-1,-1)$ then $sgn(D,C) = 1$ iff $t(C) = (1,1)$.*

Definition 2.11 *Let G be a proper g-graph. Let D be a relevant orientation of G. Let $r(D) = (r_1, ..., r_{2g})$. We let $c(r(D))$ equal to the product of c_i, $i = 0, ..., g-1$, where $c_i = c(r_{2i+1}, r_{2i+2})$ and $c(1,1) = c(1,-1) = c(-1,1) = 1/2$ and $c(-1,-1) = -1/2$.*

Observe that $c(r(D)) = (-1)^n 2^{-g}$, where $n = |\{i; r_{2i+1} = r_{2i+2} = -1\}|$.

Corollary 2.12 *Let G be a proper 1-graph. Let D_1, D_2, D_3, D_4 be the relevant orientations of G. Then*

$$\mathcal{P}(G,x) = \sum_{i=1}^{4} c(r(D_i))\mathcal{P}(D_i, M_0).$$

Corollary 2.12 holds for all proper g-graphs. In order to deduce it we start with another corollary of Theorem 2.9.

Corollary 2.13 *Let G be a proper g-graph, D a relevant orientation of G and let P be a perfect matching of G. Then $sgn(D, M_0\Delta P)$ is a function of $r(D)$ and $t(M_0\Delta P)$ only. Let us denote this function by $\sigma(r(D), t(M_0\Delta P))$.*

Lemma 2.14 *Let $r = (r_1, \dots, r_{2g})$ and $t = (t_1, \dots, t_{2g})$ where $r_i, t_i \in \{1,-1\}$. Let $r^j = (r_{2j+1}, r_{2j+2})$ and $t^j = (t_{2j+1}, t_{2j+2})$, $j = 0, \dots, g-1$. Then*

$$\sigma(r,t) = \prod_{j=0}^{g-1} \sigma(r^j, t^j).$$

Proof. By Corollary 2.13, $\sigma(r,t) = sgn(D,\mathcal{C})$ where $r(D) = r$, $t(\mathcal{C}) = t$ and D is a relevant orientation of a proper g-graph G such that $G = C_0 \cup \mathcal{C}$. $\mathcal{C}$ is a set of vertex-disjoint cycles $C_1, \dots, C_k$ with the following properties:

1. each C_i is alternating with respect to a perfect matching M_0 of G_0,

2. for each i, j $|E^i_j| \leq 1$,

3. for each i there is at most one j such that $|C_j \cap (E^i_1 \cup E^i_2)| \geq 1$,

4. for each C_j there is exactly one i such that C_j intersects $E^i_1 \cup E^i_2$.

Hence, $\sigma(r,t) = sgn(D,\mathcal{C}) = \prod_{i=1}^{k} sgn(D, C_i) = \prod_{i=1}^{k} sgn(D^i, C_i)$

where D^i is the restriction of D to $C_0 \cup C_i$. Finally observe that, by Corollary 2.10, $\sigma(z_1, z_2) = 1$ if $z_2 = (1,1)$. Hence, using Corollary 2.13, we have that $\prod_{i=1}^{k} sgn(D^i, C_i) = \prod_{j=0}^{g-1} \sigma(r^j, t^j)$, which we needed to prove. □

Theorem 2.15 *Let G be a proper g-graph. Then*

$$\mathcal{P}(G,x) = \mathcal{L}_g(G,x) = \sum_{i=1}^{4^g} c(r(D_i))\mathcal{P}(D_i, M_0)$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Proof. Let P be a perfect matching of G. In each $\mathcal{P}(D_i, M_0)$, $x(P)$ has coefficient equal to $sgn(D_i, M_0 \Delta P)$. By Corollary 2.13, $sgn(D_i, M_0\Delta P) = \sigma(r(D_i), t(M_0\Delta P))$.

Let

$$\mathcal{K}_g(t(M_0\Delta P) = \sum_{i=1}^{4^g} c(r(D_i))\sigma(r(D_i), t(M_0\Delta P))$$

.

Theorem follows from the following claim:

Claim. $\mathcal{K}_g(t(M_0\Delta P)) = 1$ for each $t(M_0\Delta P)$.

Proof of Claim. We proceed by induction on g.

The basis of the induction when $g = 1$ is Corollary 2.12.

To prove the induction step we introduce the following notation. If z is a vector in $\{1,-1\}^{2g}$ then we let $z = (z(0), \ldots, z(g-1))$ where $z(i) = (z_{2i+1}, z_{2i+2})$.

We call two relevant orientations D, D' of G *equivalent* if $(r(D)(1), \ldots, r(D)(g-1)) = (r(D')(1), \ldots, r(D')(g-1))$. Clearly, the equivalence classes consist of 4 elements; let $\mathcal{R}_1, \ldots, \mathcal{R}_{4^{g-1}}$ be the equivalence classes of the relevant orientations of G and let $\mathcal{R}_j = \{D^j_1, D^j_2, D^j_3, D^j_4\}$, $j = 1, \ldots, 4^{g-1}$.

Finally let $r(D^j_i)(k) = r^j_i(k)$, $k = 0, \ldots, g-1$ and $t = t(M_0\Delta P)$. We have that

$$\mathcal{K}_g(t) = \sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r(D^j_i))\sigma(r(D^j_i), t).$$

Now, by Lemma 2.14, this equals to

$$\sum_{j=1}^{4^{g-1}} \sum_{i=1}^{4} c(r_i^j(0)) c(r_i^j(1), \ldots, r_i^j(g-1)) \prod_{k=0}^{g-1} \sigma(r_i^j(k), t(k)).$$

By the definition of the equivalence classes, $r_1^j(k) = r_2^j(k) = r_3^j(k) = r_4^j(k)$ for $k \geq 1$ and $j = 1, \ldots, 4^{g-1}$. Hence, we let $r_i^j(k) = r^j(k)$ and write the above summation as:

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \ldots, r^j(g-1)) \prod_{k=1}^{g-1} \sigma(r^j(k), t(k)) \sum_{i=1}^{4} c(r_i^j(0)) \sigma(r_i^j(0), t(0))$$

.

The internal sum equals to 1 for each $j = 1, \ldots, 4^{g-1}$ by the basis step of the induction, and hence, using Lemma 2.14 for the first sum, we can write the above summation as

$$\sum_{j=1}^{4^{g-1}} c(r^j(1), \ldots, r^j(g-1)) \sigma((r^j(1), \ldots, r^j(g-1)), (t(1), \ldots, t(g-1))) =$$

$$\mathcal{K}_{g-1}(t(1), \ldots, t(g-1)) = 1,$$

by the induction hypothesis for $g - 1$.

□

As a consequence of Theorem 1.4 and Theorem 2.15, we get

Corollary 2.16 *Let G be a proper g-graph. Then $s(D_i, M_0) = s(D_j, M_0)$ for each $i, j \in \{1, \ldots, 4^g\}$ and*

$$\mathcal{P}(G, x) = \mathcal{L}'_g(G, x) = s(D_1, M_0) \sum_{i=1}^{4^g} c(r(D_i)) Pf(A(D_i))$$

where D_i, $i = 1, \ldots, 4^g$, are the relevant orientations of G.

Corollary 2.17 *Let G be a graph embeddable on an orientable surface of genus g. Then $\mathcal{P}(G, x)$ may be expressed as a linear combination of 4^g Pfaffians of matrices $A(D)$, where each D is an orientation of G.*

Proof. Orientable surface O_g of genus g may be obtained from S_g as follows: for each bridge B, glue together the two segments in which B intersects the boundary of B_0, and delete B.

If a graph G is embeddable on an orientable surface O_g of genus g, then without loss of generality no vertex belongs to the boundary of B_0. Then, splitting the boundaries back to the bridges, this gives a drawing of the graph on S_g such that each vertex lies on B_0 but some edges may use several bridges.

We construct a graph G' so that we replace each edge e which uses k bridges, $k \geq 1$, by a path P_e of edges $(e_1, \ldots, e_{2k+1})$ and vertices $v_1, \ldots, v_{2k}$. We let $x'_{e_1} = x_e$ and $x'_{e_i} = 1$ for each $i > 1$. Next we add edges so that the outer face of the planar part is a cycle. We let $x'_e = 0$ for each such edge e.

Note that G' is a proper g-graph and $\mathcal{P}(G', x') = \mathcal{P}(G, x)$.

By Theorem 2.15, $\mathcal{P}(G', x')$ may be written as a linear combination of 4^g Pfaffians of $A(D')$, where each D' is a relevant orientation of G'.

Claim. For each relevant orientation D' of G' there is an orientation D of G so that $Pf(A(D')) = Pf(A(D))$ or $Pf(A(D')) = -Pf(A(D))$.

Proof of Claim. We construct D from D' in two steps:

1. delete the edges e of $G' - G$ with $x'_e = 0$,

2. for each edge e of G which was changed into a path P_e of odd length in the construction of G', orient e in the direction in which an odd number of edges of P_e is directed in D': this is uniquelly determined since P_e has odd length.

If P is a perfect matching of G then there is a unique perfect matching P' of G' such that $x(P) = x'(P')$.

Observe that $sgn(D, P\Delta Q) = sgn(D', P'\Delta Q')$ for each pair of perfect matchings P, Q of G. The claim now follows from Theorem 1.4.

End of Claim.

This finishes the proof of the Corollary.

$\square$

3 Exact Matching, Pfaffian Orientation, and Permanents.

The results of the previous sections have interesting algorithmic consequences.

Theorem 3.1 *Let S be an orientable surface and k a fixed positive integer. Let $\mathcal{G}$ be the class of graphs which may be embedded on S and such that the edges are partitioned into at most k classes and variables x_e are equal in each class. Then $\mathcal{P}(G, x)$ may be determined efficiently for $G \in \mathcal{G}$. As a consequence, it is possible to verify efficiently whether $G \in \mathcal{G}$ is Pfaffian.*

Proof. It follows from Theorems 2.15 and 2.17 that $\mathcal{P}(G,x)$ may be determined efficiently.

Concerning the problem of recognizing whether G is Pfaffian, we procede as follows: it was proved by Vazirani and Yannakakis (see the proof of theorem 3.1 of [13]) that there is an orientation D of graph G so that G is Pfaffian if and only if D is its Pfaffian orientation, and moreover D may be constructed efficiently. Hence $Pf(A(D))$ equals to the number of the perfect matchings of G if and only if G is Pfaffian, and it means that we can decide efficiently whether a graph is Pfaffian once we can compute efficiently its number of perfect matchings.

□

In particular the following well-known problem may be solved efficiently for the graphs embeddable on an arbitrary fixed orientable surface:

Exact Matching Problem. Given a graph G with some edges coloured red, and a number h. Find out whether G has a perfect matching with exactly h red edges.

Next we consider permanents of square matrices.

In 1913, Polya ([9]) suggested computing the permanent of a matrix A by changing the signs of some entries of A so that the determinant of the resulting matrix equals the permanent of A. Let us call a $0,1$-matrix A *convertible* if such a change is possible.

Szego ([11]) pointed out in the same year that not all matrices are convertible.

This may be explained nowadays using a complexity argument. There is an efficient algorithm to compute the determinant, while Valiant proved that the problem of computing the permanent of a $0,1$ matrix is $\#P$-complete (see [12]).

The computational problem of recognition of convertible matrices has been proved recently to admit a polynomial algorithm by McCuaig, Robertson, Seymour and Thomas (see [8]). An earlier paper of Galluccio and Loebl (see [3]) contains a related algorithmic result, as well as the history of the problem.

The recognition of convertible matrices is equivalent to the problem of recognition of bipartite Pfaffian graphs, and to the 'Even Cycle Problem': given a directed graph, decide whether it contains a directed cycle of even length.

Let A be a square matrix. Denote by $G(A)$ the bipartite graph whose two bipartition classes are indexed by the rows and the columns of A, and for each edge ij, $a_{ij} = x_{ij}$. Then $per(A) = \mathcal{P}(G(A), x)$.

This means that Corollary 2.17 provides a new combinatorial way to compute permanents of square matrices: $per(A)$ may be written as a linear combination of 4^g terms of form $Pf(A(D))$, where D is an orientation of $G(A)$ and g is the genus of $G(A)$.

Since $G(A)$ is a bipartite graph, the non-zero entries of $A(D)$ belong to two blocks A_1, A_2, where A_1 is obtained from A by changing the sign of some entries and $A_2 = -A_1$. Moreover $|Pf(A(D)| = |det(A_1)| = |det(A_2)|$ by Theorem 1.5.

This means that the method of Polya may be completed as follows:

Corollary 3.2 *Let A be a square matrix. Then $per(A)$ may be expressed as a linear combination of terms of form $det(A^i)$, $i = 1, ..., 4^g$, where each A^i is obtained from A by changing the sign of some entries.*

Concluding Remarks. In a continuation of this paper which is in preparation we consider the following related problems:

1. The generating function of T-joins and edge-cuts,
2. The evaluation of Tutte polynomial and the weight enumerator of a binary code,
3. A theory of crystal structures and application of the proposed method to solve the Ising problem for three-dimensional crystal structures.

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