Regular monomorphisms of Hausdorff frames

T. Plewe*, A. Pultr** and A. Tozzi

In honour of Horst Herrlich on the occassion of his 60th birthday

Abstract : Regular monomorphisms in the category of Hausdorff frames are characterized by means of a naturally defined closure operator; this is used also to characterize the epimorphisms. Further it is shown that for spatial (strongly) Hausdorff frames the regular monomorphisms do not generally coincide with the quotients, and do not generally compose. Also, an additional property (under which regular monomorphisms do compose) is briefly studied.

1991 Mathematics Subject Classification : 18A20, 18B30, 54B15, 06D10

Keywords : frames, locales, Hausdorff property, regular monomorphisms resp. epimorphisms, quotient mappings

Introduction

The aim of this paper is to contribute to the study of algebraic (point-free) representations of quotients of spaces, and to related questions concerning regular epimorphisms in the category of locales (regular monomorphisms in the category of frames). In a previous paper [**22**], a certain closure operator $\mathcal{E}$ on subframes of a frame has

---

*Financially supported by the British Science and Engineering Research Council under the "Foundational Structures in Computing Science" research project.
**The support of the Italian C.N.R. and the Grant Agency of the Czech Republic under Grant 201/96/0119 is gratefully acknowledged.

been defined, with the property that for a certain class of spaces (including a.o. all the metrizable ones), a surjective map is a quotient iff the induced embedded subframe is $\mathcal{E}$-closed. The naturally arising questions as to how far $\mathcal{E}$-closedness characterizes regularity, and whether epimorphisms in frames (monomorphisms in locales) are exactly the $\mathcal{E}$-dense morphisms, remain so far unsolved. Here we restrict ourselves to the category of Hausdorff frames and define another natural closure operator $\overline{L}^M$ for subframes $L \subseteq M$ which does have the the properties indicated: a monomorphism $h : L \to M$ is regular iff $\overline{h[L]}^M = h[L]$, and a frame homomorphism $h : L \to M$ is an epimorphism iff $\overline{h[L]}^M = M$. On the other hand, following the techniques from the recent article [**20**] it is proved that even in this narrower context the regularity does not quite characterize quotient maps: there are surjections of (strongly) Hausdorff spaces which are regular epimorphisms in the category of locales without being quotient maps, and also that even in this context regularity is not preserved under composition.

One of the reasons why regular epimorphisms in Hausdorff locales do not necessarily compose is in the fact that the closure in question does not always have a certain desirable property. This in turn is connected with the fact that (unlike in spaces, where cartesian products of surjections are, trivially, surjections) in the category of frames the (co)products $\mu \oplus \mu$ with monomorphic μ are not necessarily monomorphic. This phenomenon is discussed in the last section of the article.

For technical reasons we have decided not to be quite consistent in viewing the situation. In most of the text we keep the frame (algebraic) approach as it makes dealing with the closures easier. When dealing with spaces in Section 4, however, we have found the localic (geometric) one more appropriate.

1. Preliminaries

1.1. Recall that a *frame* is a complete lattice L satisfying the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge b \mid b \in S\}$$

for every $a \in L$ and every $S \subseteq L$, and a *frame homomorphism* $h : L \to M$ is a mapping preserving all joins (including the bottom 0) and finite meets (including the top 1). The resulting category will be denoted by

$$\mathbf{Frm}.$$

The lattice $\mathfrak{O}(X)$ of open sets of a topological space X is a frame, and if $f : X \to Y$ is a continuous map we have a frame homomorphism $\mathfrak{O}(f) : \mathfrak{O}(Y) \to \mathfrak{O}(X)$ defined by $\mathfrak{O}(f)(U) = f^{-1}(U)$. Thus one has a contravariant functor

$$\mathfrak{O} : \mathbf{Top} \to \mathbf{Frm}.$$

The dual of **Frm** is called the category of locales and denoted by **Loc**. The *covariant* $\mathfrak{O} : \mathbf{Top} \to \mathbf{Loc}$ has a right adjoint $\mathsf{pt} : \mathbf{Loc} \to \mathbf{Top}$; a frame (locale) L is said to be *spatial* if it is isomorphic to an $\mathfrak{O}(X)$ and this is the case iff $\mathfrak{O}(\mathsf{pt}(L)) \cong L$.

Restricted to the subcategory **Sob** of sober spaces, $\mathfrak{O} : \mathbf{Sob} \to \mathbf{Loc}$ is a full embedding.

The *pseudocomplement* of an $a \in L$ is

$$a^* = \bigvee\{x \mid x \wedge a\} = 0,$$

the largest element meeting a in 0. A frame L is *regular* if for each $a \in L$, $a = \bigvee\{x \mid x^* \vee a = 1\}$. Note that $\mathfrak{O}(X)$ is regular iff X is regular in the usual sense.

1.2. A *sublocale* (more precisely, sublocale homomorphism) is a frame homomorphism $h : L \to M$ which is onto. For instance, if Y is a subspace of X, we have the sublocale $(U \mapsto U \cap Y) : \mathfrak{O}(X) \to \mathfrak{O}(Y)$. In particular we will be interested in *open sublocales*, i.e. surjections of the form

$$\hat{a} = (x \mapsto x \wedge a) : L \to {\downarrow} a = \{x \mid x \geq a\}, \quad a \in L,$$

and in *closed* ones,

$$\check{a} = (x \mapsto x \vee a) : L \to {\uparrow} a = \{x \mid x \geq a\} \quad a \in L.$$

We say that a sublocale $h : L \to M$ contains a sublocale $k : L \to K$ and write $k \sqsubseteq h$ if there is a homomorpism g such that $k = g \cdot h$, and h and k are *equivalent* if $k \sqsubseteq h$ and $h \sqsubseteq k$. The sublocales of L up to equivalence constitute a complete lattice $\mathcal{S}(L)$, and a sublocale is said to be *complemented* if it is so in $\mathcal{S}(L)$.

Joins of open sublocales are open and one has $\bigsqcup \hat{a}_i = (\bigvee a_i)\hat{}$. Consequently, for each sublocale γ there is the largest open sublocale contained in γ; it will be denoted by int γ.

If $h : L \to M$ is a frame homomorphism and $\gamma : L \to K$ is a sublocale, the preimage of γ under h is defined by the pushout

$$\begin{tikzcd}
L \arrow[r, "h"] \arrow[d, "\gamma"'] & M \arrow[d, "h^{-1}(\gamma)"] \\
K \arrow[r] & h^{-1}(K)
\end{tikzcd}$$

In particular one has $h^{-1}(\hat{a}) = \widehat{h(a)}$.

A sublocale of a space does not necessarily have points. But

(1.2.1) *if X is scattered (that is, if each non-void subspace $Y \subseteq X$ contains a point y isolated in Y) then every sublocale of X is a subspace*([**19**]).

For more details on frames we refer to [**15**] or [**25**], for category theory to [**18**].

1.3. For a frame L consider

$$\mathfrak{D}L = \{U \subseteq L \mid \emptyset \neq U = {\downarrow}U = \{x \mid \exists u \in U,\ x \leq u\}\},$$

the frame of all non-void decreasing subsets of L. The coproduct $L \oplus L$ will be represented, as usual, as the subset of $\mathfrak{D}(L \times L)$ consisting of all the *saturated* sets, that is, of those sets U which satisfy

$$(a_i, b) \in U : \ i \in J \quad \Rightarrow \quad (\bigvee_J a_i, b) \in U$$

and similarly for (a, b_i). Since the premise is trivially satisfied if $J = \emptyset$, each saturated set U contains

$$\mathbf{O} = \{(0, a), (a, 0) \mid a \in L\},$$

---

24. W.J. Thron, *Lattice-equivalence of topological spaces*, Duke Math. J. **29** (1962), 671-679.
25. S.J. Vickers, *"Topology via Logic"*, Cambridge Tracts in Theor. Comp. Science No.5, Cambridge University Press, Cambridge 1989.

Till Plewe, Department of Computing, Imperial College, 180 Queen's Gate, London SW7 2AZ, UK

Aleš Pultr, KAM, MFF University Karlovy, Malostranské nám. 25, 11800 Praha 1 - Malá Strana, Czech Republic

Anna Tozzi, Dipartimento di Matematica Pura ed Applicata, Universita degli Studi di L'Aquila, 67100 L'Aquila, Italy

References

1. C.E. Aull & W.J. Thron, *Separation axioms between T_0 and T_1*, Indag. Math. **24** (1962), 26-37.
2. B. Banaschewski, *On pushing out frames*, Comment. Math. Univ. Carolinae **31** (1990), 13-21.
3. B. Banaschewski, *Singly generated frame extensions*, J. Pure App. Alg. **83** (1990), 1-21.
4. B. Banaschewski, *The duality of distributive continuous lattices*, Canad. J. Math. **32** (1980), 385-394.
5. B. Banaschewski and A. Pultr, *Variants of openness*, Appl. Categ. Structures **2** (1994), 331-350.
6. B. Banaschewski and A. Pultr, *Samuel compactification and completion of uniform frames*, Math. Proc. Cambridge Phil. Soc. **108** (1990), 63-78.
7. B. Banaschewski and A. Pultr, *Cauchy points of uniform and nearness frames*, Quaestiones Mathematicae **19 (1-2)** (1996), 101-127.
8. X. Chen, *Thesis*, McMaster University, Hamilton 1991.
9. R. Engelking, *"General Topology"*, Sigma Series in Pure Mathematics, Vol.6, Helderman Verlag, Berlin 1989.
10. K.H. Hoffmann and J.D. Lawson, *The spectral theory of distributive continuous lattices*, Trans. of the AMS **246** (1978), 285-309.
11. J.R. Isbell, *Atomless parts of spaces*, Math. Scand. **31** (1972), 5-32.
12. J.R. Isbell, *Function spaces and adjoints*, Math. Scand. **36** (1975), 317-339.
13. J.R. Isbell, *First steps in descriptive theory of locales*, Trans. Amer. Math. Soc. **327** (1991), 353-371.
14. J.R. Isbell, *Corrections to "First steps in descriptive theory of locales"*, Trans. Amer. Math Soc. **341** (1994), 462-468.
15. P.T. Johnstone, *"Stone Spaces"*, Cambridge University Press, Cambridge 1982.
16. A. Joyal and M. Tierney, *An extension of the Galois Theory of Grothendieck*, Memoirs of the AMS **51**, No. 309 (September 1984).
17. K. Kuratowski, *"Topology", vol.I,*, Academic Press, New York, 1966.
18. S. MacLane, "Categories for the Working Mathematician," Springer-Verlag, New York, 1971.
19. S.B. Niefield and K.I. Rosenthal, *Spatial sublocales and essential primes*, Top. Appl. **26** (1987), 263-269.
20. T. Plewe, *Quotient maps of locales*, preprint, 1996.
21. T. Plewe, *Localic products of spaces*, Proc. London Math. soc. **73** (1996), 642-678.
22. A.Pultr and A.Tozzi, *Equationally closed subframes and representation of quotient spaces* , Cahiers de Top. et Géom. Diff. Cat **XXXIV-3** (1993), 167-183.
23. A. Pultr and A. Tozzi, *Separation axioms and frame representation of some topological facts*, Appl. Categ. Structures **2** (1994), 107-118.

and $\mathbf{O}$ is the zero of $L \oplus L$. In particular we have the saturated sets

$$a \oplus b = \downarrow(a, b) \cup \mathbf{O}$$

and for each saturated set U one has

$$U = \bigvee\{a \oplus b \mid a \oplus b \leq U\} = \bigvee\{a \oplus b \mid (a, b) \in U\}.$$

The coproduct injections $\iota_i : L \to L \oplus L$ are defined by $\iota_1(a) = a \oplus 1$, $\iota_2(a) = 1 \oplus a$ so that $a \oplus b = \iota_1(a) \wedge \iota_2(b)$. Consequently, the *codiagonal* $\nabla : L \oplus L \to L$ is given by the formula

$$\nabla(U) = \bigvee\{a \wedge b \mid (a, b) \in U\}.$$

1.4. A particular role will be played by the saturated set

$$c_L = \{(u, v) \mid u \wedge v = 0\} = \{u \oplus v \mid u \wedge v = 0\} \in L \oplus L.$$

Note that

$$\check{c}_L : L \oplus L \to \uparrow c_L$$

is the *closure* of the codiagonal ∇, that is, the smallest closed sublocale containing ∇.

1.5. A frame is said to be *Hausdorff* (in the sense of Isbell, [**11**]) if the codiagonal in L is closed, that is, if there is an isomorphism α making

$$\begin{tikzcd} L \oplus L \arrow[r, "\nabla"] \arrow[dr, "\check{c}_L"] & L \arrow[d, "\alpha"] \\ & \uparrow c_L \end{tikzcd}$$

commute. (If such an α exists it is necessarily the inverse of the frame homomorphism $\uparrow c_L \ni \bigvee a_i \oplus b_i \mapsto \bigvee a_i \wedge b_i \in L$.) Each regular frame is Hausdorff (see [**15**]). For T_0-spaces X, the usual Hausdorff property is implied by $\mathfrak{O}(X)$ being Hausdorff, but the converse is not generally true. Hence, we will sometimes use the expression "strongly Hausdorff" to indicate the fact that also $\mathfrak{O}(X)$ is Hausdorff.

1.6. The following theorem was proved, first, by B. Banaschewski for the regular case in [**2**]. The Hausdorff case appeared in the Xiangdong Chen's Thesis [**8**]. This may be not easily accessible; as the proof is short we will present it.

THEOREM. *Let L be a Hausdorff frame. Then the coequalizer of $f, g : L \to M$ is the $\check{c} : M \to {\uparrow}c$ where*

$$c = \bigvee \{f(x) \wedge g(y) \mid x \wedge y = 0\}.$$

PROOF: We will use the notation of 1.4 and 1.5. Obviously, $c = \nabla(f \oplus g)(c_L)$. We have

$$(a \oplus 1) \vee c_L = \alpha\nabla(a \oplus 1) = \alpha(a) = \alpha\nabla(1 \oplus a) = (1 \oplus a) \vee c_L$$

and hence

$$f(a) \vee c = \nabla(f \oplus g)((a \oplus 1) \vee c_L) = \nabla(f \oplus g)((1 \oplus a) \vee c_L) = g(a) \vee c.$$

On the other hand, if $\varphi f = \varphi g = h$ for a $\varphi : M \to K$, we have

$$\varphi(c) = \bigvee \{h(x \wedge y) \mid x \wedge y = 0\} = 0$$

and hence we can define $\overline{\varphi} : {\uparrow}c \to K$ by $\overline{\varphi}(x) = \varphi(x)$. □

1.7. Recall from ([**1**]) that a space X satisfies the separation axiom T_D if for each $x \in X$ there is an open U such that $x \in U$ and $U \smallsetminus \{x\}$ is open. T_D sits between T_0 and T_1. For us, the following easy fact will be important:

(1.7.1) *If Y is a T_D-space then a continuous $f : X \to Y$ is onto iff $\mathfrak{O}(f)$ is one-one*

(for the role of T_D in algebraic representation of spaces and mappings see [**24**], [**23**], [**5**]).

A spatial frame will be called a T_D frame if it can be represented as $\mathfrak{O}(X)$ with a T_D-space X.

---

reason for this is that $\coprod_{q\in\mathbb{Q}} \mathbb{Q}_q$ is scattered and hence has spatial localic square, but $\mathbb{Q} \times \mathbb{Q}$ is not spatial so $f \times f$ cannot be surjective (see [**21**] for results on spatiality of products of spatial locales).

5.9. For any $L \subseteq M$ we obviously have

$${\downarrow}^{M\times M} c_L \le c^L_M \le c_M.$$

As $c_M = c^M_M$, we have in the maximum case of $c^L_M = c_M$ the maximum closure $\overline{L}^M = M$ (recall 2.2.(4)). One would be tempted to expect also in the minimum case of $c^M_L = {\downarrow}c_L$ the minimum closure $\overline{L}^M = L$. The example 5.8 shows that this is not the case. We indeed have $c^M_L = {\downarrow}c_L$ but we cannot have $\overline{L}^M = L$: by 3.6, the embedding $L \subseteq K$ is not a regular monomorphism since the identity carried $(X, L) \to (X, K)$ is not a quotient.

5.10. By the Proposition in 5.8 we also see that a category of locales, or enriched locales, containing all the separable metrizable ones and such that the products coincide with (or are carried by) the products in **Loc**, as for instance the categories of

- locales, regular locales, completely regular locales, or paracompact locales, with all localic maps, or
- uniform locales, or nearness locales ([**11**],[**6**],[**7**]) with uniform locallic maps,

cannot be cartesian closed.

Acknowledgement : Stimulating discussions the second author had on the topic with Professors D. Pumplün and R. Börger during the visit at the FernUniversität Hagen supported by the Alexander von Humboldt-Stiftung are gratefully acknowledged.

5.7. LEMMA. *Let L, K be topologies on the same set X, $L \subseteq K$. Let for each $u \in K$ there be a $\widetilde{u} \in L$ such that u is contained and dense in $\widetilde{u}$ in (X, L). Then*

$$c_L^K = \downarrow^{K\times K} c_L.$$

PROOF: Let $(a_i, b) \in \downarrow c_L$. Then $(a_i, b) \le (u_i, v_i) \in c_L$. As $u_i \cap b = 0$, we have, by density, $u_i \cap \widetilde{b} = 0$. Hence $(a_i, b) \le (u_i, \widetilde{b}) \in c_L$ and $(\bigvee a_i, b) \le (\bigvee u_i, \widetilde{b})$. Thus, $\downarrow c_L$ is saturated in $K \times K$. □

5.8. The example in 4.6 was not regular, and we are so far unable to produce a regular one. We will see now, however, that homomorphisms between very well behaved (even metric) spatial frames can fail to be bi-monomorphic.

PROPOSITION. *Let L be the standard topology on the set $\mathbb{R}$ of reals and let M be the discrete topology on the same set. There are frame inclusions $L \subseteq K \subseteq M$ such that $(\mathbb{R}, K)$ is a separable metric space, and $k : K \subseteq M$ is not a bi-monomorphism (while $h : L \subseteq M$ is).*

PROOF: Let K be the L augmented by the set of all rational numbers $\mathfrak{R}$ and by the set of irrationals $\mathfrak{I}$. Thus, K is constituted by the

$$(u \cap \mathfrak{R}) \cup (v \cap \mathfrak{I}), \quad u, v \in L$$

and each such set is dense in the $u \cup v$ so that $L \subseteq K$ satisfies the conditions of 5.7. Furthermore, K is separable metric since obviously it is homeomorphic to the disjoint sum of $\mathfrak{R}$ and $\mathfrak{I}$ equipped with the standard topologies. Set $k : K \subseteq M$. By 5.6 we have

$$c_L^M = c_M.$$

Thus, $c_L^M \cap (K \times K) = c_K$. On the other hand, by 5.7, $c_L^K = \downarrow c_L \neq c_K$ since $(\mathfrak{R}, \mathfrak{I})$ is evidently not in $\downarrow c_L$. □

The map $f : \coprod_{q\in\mathbb{Q}} \mathbb{Q}_q \to \mathbb{Q}$ where $\mathbb{Q}_q$ is constructed as in 4.4 provides another (simple example of a quotient map of separable metrizable spaces for which the map $f^* \oplus f^*$ fails to be bi-moomorphic. The

A sober T_D-space X has the largest pointless sublocale $\mathbf{pl}(X)$ which can be computed in $\mathcal{S}(\mathfrak{O}(X))$ either as the join of all the pointless sublocales, or as the intersection of all complements of scattered subspaces of X ([**13**], [**14**]).

1.8. We will also need the following two special properties of spaces from [**22**] :
A space is said to satisfy (ApC) if

> *for each non-open $M \subseteq X$ there are a closed F and open U, V such that*
>
> $$U \cup (M \cap V) = U \cup (F \cap V)$$
>
> *and that this set is not open.*

It is said to satisfy (ApP) if

> *for any $M \subseteq X$ and any $x \in M \smallsetminus \mathrm{int}M$ there is a $C \subseteq X$ such that*
> $$C \cap M = \emptyset \quad \textit{and} \quad \overline{C} \cap M = \{x\}.$$

Each metrizable space satisfies (ApP) and for each T_D-space (ApP) implies (ApC).

2. A closure operator and a characterization of regular monomorphisms and epimorphisms

2.1. Let L be a subframe of M, let $j : L \subseteq M$ be the embedding homomorphism. Put

$$c_L^M = \bigvee\{x \oplus y \in M \oplus M \mid x, y \in L,\ x \wedge y = 0\}.$$

Thus (recall 1.3)

(2.1.1) *c_L^M is equal to $(j \oplus j)(c_L)$, the smallest saturated element of $\mathfrak{D}(M \times M)$ containing c_L.*

From 1.6 we immediately see that

(2.1.2) *if L is Hausdorff then $\check{c}_L^M$ is the coequalizer of $\iota_1 j$, $\iota_2 j$.*

Obviously,

(2.1.2) *if* $L \subseteq K \subseteq M$ *then* $c^M_L \leq c^M_K$.

2.2. Notation as in 2.1. We set

$$\overline{L}^M = \{a \in M \mid (a \oplus 1) \vee c^M_L = (1 \oplus a) \vee c^M_L\}.$$

Thus,

(2.2.1) $\overline{L}^M \subseteq M$ *is the equalizer of* $\check{c}^M_L \iota_1$ *and* $\check{c}^M_L \iota_2$.

We have

PROPOSITION. *(1)* $\overline{L}^M$ *is a subframe of* M*,*

(2) $\overline{\{0,1\}}^M = \{0,1\}$,

(3) $L \subseteq K \quad \Rightarrow \quad \overline{L}^M \subseteq \overline{K}^M$,

(4) if L *is Hausdorff then* $L \subseteq \overline{L}^M$,

(5) if L *is Hausdorff then* $\overline{\overline{L}^M}^M = \overline{L}^M$.

PROOF: (1) follows from (2.2.1), (3) from (2.1.2) and (4) from (2.1.2).

(2) : As $c^M_{\{0,1\}} = \mathbf{O}$, we have to have $a \oplus 1 = 1 \oplus a$ which (recall 1.3.1) holds only for $a = 0, 1$.

(5) : Put $K = \overline{L}^M$. Thus, we must prove that $c^M_K \leq c^M_L$. Suppose $x, y \in K$ and $x \wedge y = 0$. Then

$$\begin{aligned}(x \oplus y) \vee c^M_L &= ((x \oplus 1) \wedge (1 \oplus y)) \vee c^M_L = \\ &= ((x \oplus 1) \wedge (y \oplus 1)) \vee c^M_L = ((x \wedge y) \oplus 1) \vee c^M_L = c^M_L,\end{aligned}$$

that is, $x \oplus y \leq c^M_L$. ☐

2.3. THEOREM. *Let* L *be a Hausdorff frame. Then a one-one homomorphism* $h : L \to M$ *is a regular monomorphism iff* $\overline{h[L]}^M = h[L]$.

PROOF: A monomorphism $h : L \to M$ is regular iff it is the equalizer of the f, g from the pushout

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "h"'] & M \arrow[d, "f"] \\ M \arrow[r, "g"] & P \end{tikzcd}$$

5.5. LEMMA. *Let* $k : L \subseteq M$ *be a bi-monomorphism,* L, M *Hausdorff. Then for any subframe* $L \subseteq K$*,*

$$\overline{L}^M \cap K = \overline{L}^K.$$

PROOF: Obviously, for $x, y \in K$ we have $(x \oplus_M y) \vee c^M_L = (k \oplus k)((x \oplus_K y) \vee c^K_L)$. Consequently, for $a \in K$ we have

$$(a \oplus_M 1) \vee c^M_L = (1 \oplus_M a) \vee c^M_L \quad \text{iff} \quad (a \oplus_K 1) \vee c^K_L = (1 \oplus_K a) \vee c^K_L.$$

☐

THEOREM. *Let* L *be Hausdorff, let* $h : L \to K$ *be a regular monomorphism and let* $k : K \to M$ *be a regular bi-monomorphism.Then* $k \cdot h$ *is a regular monomorphism.*

PROOF: Think of the h and k as of actual embeddings. If $\overline{L}^K = L$ and $\overline{K}^M = K$ we have $\overline{L}^M \subseteq \overline{K}^M = K$ and hence $\overline{L}^M \cap K = \overline{L}^K = L$. ☐

COROLLARY. *Let* $h : L \subseteq M$ *be an extremal monomorphism which is not regular,* L, M *Hausdorff. Then* $k : \overline{L}^M \subseteq M$ *is not a bi-monomorphism.*

PROOF: Put $K = \overline{L}^M$. As h is not regular, $L \neq K$. Suppose k is bi-monomorphic. Then $\overline{L}^K = \overline{L}^M \cap K = K$ and hence $L \subseteq K$ is a non-trivial epimorphism contradicting the extremality. ☐

5.6. LEMMA. *Let* L *be a regular* T_1 *topology on a set* X *and let* M *be the discrete topology on the same set (that is,* $M = \exp X$*). Then*

$$c^M_L = c_M.$$

PROOF: For $x \in X$ consider a system of open neighbourhoods u_i, $i \in J$, such that $\bigcap \overline{u}_i = \{x\}$. As $(V \smallsetminus \overline{u}_i, \{x\}) \leq (X \smallsetminus \overline{u}_i, u_i)$, the first couple is in $\downarrow c_L \subseteq c^M_L$. As $\bigcup (X \smallsetminus \overline{u}_i) = X \smallsetminus \{x\}$ we have $(X \smallsetminus \{x\}, \{x\}) \in c^M_L$. Consequently, $(A, \{x\}) \in c^M_L$ for any A not containing x and finally, if $A \cap B = \emptyset$, $(A, B = \bigcup \{\{x\} \mid x \in B\}) \in c^M_L$. ☐

5. Bi-monomorphic frame homomorphisms

5.1. Let us return to the frame notation. By 4.4 we have that

> *if $h : L \to K$ is a regular monomorphism between spatial frames which is a quotient of spaces, then for any regular monomorphism $k : K \to M$, $k \cdot h$ is again regular.*

In this section we will present a class of regular monomorphisms $k : K \to M$ composing to regular monomorphisms with all regular $h : L \to K$.

5.2. A frame homomorphism $h : L \to M$ will be called *bi-monomorphism* if $h \oplus h : L \oplus L \to M \oplus M$ is a monomorphism. Obviously, as $\iota_1 h = (h \oplus h)\iota_1$, each bi-monomorphism is a monomorphism.

PROPOSITION. *$h : L \subseteq M$ is a bi-monomorphism iff for each saturated $U \subseteq L \times L$ one has $(h \oplus h)(U) \cap (L \times L) = U$.*

PROOF: A frame homomorphism φ is a monomorphism iff $\varphi_+\varphi$=id for φ_+ the right Galois adjoint of γ. As it is easy to see that $(h \oplus h)_+ = (V \mapsto V \cap (L \times L))$, the statement follows. $\square$

5.3. In **Top**, if f is a surjection than $f \times f$ is one as well. Since $\mathfrak{O}$ preserves finite products of Čech-complete ($\equiv$ absolutely G_δ) spaces ([**12**], [**10**], [**4**]), this fact yields

PROPOSITION. *Let L be a Čech-complete (in particular, locally compact) T_D-frame. Let M be spatial. Then each monomorphism $h : L \to M$ is a bi-monomorphism.*

5.4. OBSERVATION. *Let a subframe $L \subseteq M$ be closed under all meets. Then the embedding $h : L \subseteq M$ is a bi-monomorphism. Moreover, for any subframe K of M with $L \subseteq K \subseteq M$ one has $c_L^M \cap (K \times K) = c_L^K$.*

(Indeed, in this case obviously $(h \oplus h)(U) = {\downarrow}^{M \times M} U$ for any saturated U, hence $c_L^M = {\downarrow}^{M\times M} c_L$ and $c_L^K = {\downarrow}^{K \times K} c_L$.)

COROLLARY. *Each open monomorphism $L \to M$, in particular each monomorphism with L complete Boolean, is a bi-monomorphism.*

Now the pushout can be constructed with $f = \gamma \cdot \iota_1$, $g = \gamma \cdot \iota_2$ where $\gamma = \mathrm{coequ}(\iota_1 h, \iota_2 h)$. By (2.1.2), $\gamma = \check{c}^M_{h[L]}$ and by (2.2.1) the equalizer of $\check{c}^M_{h[L]}\iota_1$ and $\check{c}^M_{h[L]}\iota_2$ is the embedding $\overline{h[L]}_M \subseteq M$. $\square$

2.4. It is a general fact (see [**20**]) that each open monomorphism in the category of frames is regular. Using 2.3 one can present a very simple proof for the Hausdorff case: Represent h as an embedding $L \subseteq M$. If h is open, L is a complete Heyting subalgebra of M ([**16**], see also [**5**]). For an $a \in \overline{L}$ consider

$$U = \{(r \vee s, b) \mid r \leq a,\ (s, b) \leq (x, y) \in c_L\} \subseteq M \times M.$$

It is easy to check that U is saturated. Considering $s = 0$ and $(x, y) = (0, 1)$ we see that $a \oplus 1 \subseteq U$. Obviously $c_L \subseteq U$. Thus, there are $r, s \in M$ and $x, y \in L$ such that

$$r \leq a,\ r \vee s = 1,\ s \leq x,\ a \leq y \quad \text{and} \quad x \wedge y = 0.$$

Hence, $a \vee x = 1$ and $a \wedge x = 0$ so that $a = x^* \in L$.

Note : If L is regular (more generally, if it is *fit* - see [**11**]), every meet preserving frame homomorphism is open. Thus,

> *in the category of regular frames, every monomorphism which preserves all meets is regular.*

This statement has obviously no counterpart in the category of all frames.

As Hausdorff frames are not necesarily fit we cannot conclude that just preserving meets suffice in the more general Hausdorff case; of course, even here we use just preserving meets and pseudocomplements, not the general Heyting operation.

2.5. LEMMA. *Let L be a subframe of M. Let $f, g : M \to N$ coincide on L. Then they coincide on $\overline{L}^M$.*

PROOF: We have $\nabla(f \oplus g)(c_L^M) = \bigvee\{f(x \wedge y) \mid x \wedge y = 0,\ x, y \in L\} = 0$, and $\nabla(f \oplus g)(a \oplus 1) = f(a)$, $\nabla(f \oplus g)(1 \oplus a) = g(a)$. If $(a \oplus 1) \vee c_L^M = (1 \oplus a) \vee c_L^M$ we obtain

$$f(a) = \nabla(f \oplus g)((a \oplus 1) \vee c_L^M) = \nabla(f \oplus g)((1 \oplus a) \vee c_L^M) = g(a). \qquad \square$$

2.6. THEOREM. *Let L be a Hausdorff frame. Then a morphism $h : L \to M$ is an epimorphism iff $\overline{h[L]}^M = M$.*

PROOF: If $\overline{h[L]}^M = M$, h is an epimorphism by 2.5. On the other hand, the embedding $j : \overline{h[L]}^M \subseteq M$ is the equalizer of $\alpha = \check{c}^M_{h[L]} \cdot \iota_1$ and $\beta = \check{c}^M_{h[L]} \cdot \iota_2$ (recall (2.2.1)). Thus, if j is not onto, $\alpha \neq \beta$. As $h = jg$ for a $g : L \to \overline{h[L]}^M$, we have $\alpha h = \alpha j g = \beta j g = \beta h$ and hence h is not an epimorphism. □

3. Regular monomorphisms and quotients of spaces. The relation to equational closure

3.1. Another closure operator was defined in [**22**] : If L is a subframe of M,

$$\mathcal{E}^M(L)$$

is the least subframe K of M containing L such that whenever $a \vee x$, $a \wedge x$ and a are in K, also x is in K. In [**22**] A. Pultr and A. Tozzi proved

THEOREM. *Let $p : X \to Y$ be continuous, let Y satisfy T_D and (ApC). Set $L = \mathfrak{O}(Y)$, $M = \mathfrak{O}(X)$ and $h = \mathfrak{O}(p)$. Then p is a quotient mapping iff $\mathcal{E}^M(h[L]) = h[L]$.*

3.2. For a continuous mapping $p : X \to Y$ which is onto define

$$C_p^X = \bigcup \{p^{-1}(U) \times p^{-1}(V) \mid U, V \in \mathfrak{O}(Y),\ U \cap V = \emptyset\}.$$

More generally, if L is a subframe of $\mathfrak{O}(X)$ we put

$$C_L^X = \bigcup \{U \times V \mid U, V \in L,\ U \cap V = \emptyset\}.$$

Set

$$\mathcal{F}^X(L) = \{W \in \mathfrak{O}(X) \mid (W \times X) \cup C_L^X = (X \times W) \cup C_L^X\}.$$

THEOREM. *There exist regular epimorphisms between spatial Hausdorff locales which are not quotient maps.*

PROOF: Partition the unit interval $\mathbb{I}$ into two Bernstein sets B_1 and B_2. Put $X = \mathbb{I}_{sd}$. Then X is strongly Hausdorff. Denote the subspaces of X induced by B_1 and B_2 by X_1, resp. X_2. (So $X_1 \wedge X_2 = \mathsf{pl}(X)$.) Consider the map $f : X_1 + X_2 \to X$ induced by the respective subspace inclusions. It is a regular epimorphism between spatial Hausdorff locales (that $X_1 + X_2$ and X are Hausdorff follows from the preceding proposition).

To see that f is a regular epimorphism consider its kernel pair

$$\begin{tikzcd} X_1 + \mathsf{pl}(X) + \mathsf{pl}(X) + X_2 \simeq (X_1 + X_2) \times_X (X_1 + X_2) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X. \end{tikzcd}$$

Let $U = U_1 + U_2$ be an open sublocale of $X_1 + X_2$. Denote the images of U, U_1 and U_2 under f by V, V_1 and V_2, respectively. U is equipped with descent data relative to f if and only if $\mathsf{pl}(V_1) = \mathsf{pl}(V_2)$. Assume that this is the case. There exist open subspaces W_i of X such that $V_i = V_i \wedge W_i$. Put $R_{i,j} = W_i \wedge X_j$. Then $R_{i,j} = V_i$ and $\mathsf{pl}(R_{i,3-i}) = \mathsf{pl}(R_{i,i})$. Since V_1 and V_2 also have identical pointless parts that implies that $R_{i,3-i} \smallsetminus V_{3-i}$ is scattered and hence closed. Subtracting this scattered subspace from W_i gives therefore another open subspace W'_i of X which is contained in V and still intersects X_i in V_i. Hence $V = W'_1 \wedge W'_2$ is open. That $f^{-1}(V) = U$ follows immediately.

Finally, $f : X_1 + X_2 \to X$ is clearly not a quotient map because $f^{-1}(X_i)$ in **Top** is open without X_i being open. □

Using 4.3 we immediately get

COROLLARY. *Regular epimorphisms between spatial Hausdorff locales are not closed under composition.*

The space X used in the theorem above fails to be regular by a wide margin: any sequence $\{a_n\}_{n \in N}$ which converges in $\mathbb{I}$ to a point $a \notin \{a_n \mid n \in N\}$ is closed in X but canot be separated from a by open sets in X. So this construction cannot be used to obtain regular epimorphisms of regular spatial locales which are not quotient maps.

on $\mathsf{pt}(X_{sd}) = \mathsf{pt}(X)$.) Since $\mathfrak{O}(X_{sd})$ is generated by the union of the images of the inclusions $\mathfrak{O}(X_{\{S_1,\dots,S_n\}}) \to \mathfrak{O}(X_{sd})$ it is also generated by open sublocales $U \smallsetminus S$. We need to show that the covering relations for $\mathfrak{O}(X_{sd})$ and for $\mathcal{T}$ coincide. So consider $U \smallsetminus S$ in $\mathfrak{O}(X_{sd})$ and a family $\{U_i \smallsetminus S_i \mid i \in I\}$ such that $\mathsf{pt}(U \smallsetminus S) \le \bigcup \mathsf{pt}(U_i \smallsetminus S_i)$. We may assume that $U = \bigvee U_i$. It suffices to show that $U \smallsetminus S \le \bigvee (U_i \smallsetminus S_i)$ holds locally, i.e. that for all $j \in I$,

$$(*) \qquad U_j \smallsetminus S \le \bigvee_{i\in I} ((U_j \wedge U_i) \smallsetminus S_i).$$

Since

$$\begin{aligned}\bigvee_{i\in I} ((U_j \wedge U_i) \smallsetminus S_i) &= \bigvee_{i\in I} (((U_j \wedge U_i) \smallsetminus S_i) \vee (U_j \smallsetminus S_j)) = \\ &= \bigvee_{i\in I} (U_j \smallsetminus (S_j \smallsetminus (U_i \smallsetminus S_i))),\end{aligned}$$

we see that it suffices to show that the restriction of $(*)$ to $P = (U_j \smallsetminus S) \smallsetminus (U_j \smallsetminus (S \vee S_J))$ holds. Since P is scattered and locally closed in $X_{\{S,S_j\}}$, P is a discrete subspace of X_{sd}. So we are done because $\mathsf{pt}(\bigvee_{i\in I} ((U_j \wedge U_i) \smallsetminus S_i) \ge \mathsf{pt}(U_j \smallsetminus S)$. ☐

4.6. Recall that a Bernstein set in a space X is a subspace B which meets every perfect subspace of X but contains none. (Using the axiom of choice one can construct Bernstein sets in each dense-in-itself completely metrizable space [**17**],[**9**].) If B is a Bernstein set in X then so is its set theoretic complement B'; furthermore $B \wedge B' = \mathsf{pl}(X)$. This last property makes Bernstein sets a useful tool for the construction of examples involving spatial locales for which the topological and the localic behaviour differ (see e.g. [**21**]), because among pairs of disjoint subspaces of X, pairs B, B' as above have maximal localic intersection. The following theorem is a strengthening of a result which appeared in [**20**]; there the spaces were not required to be strongly Hausdorff. Both proofs run along similar lines.

(Note the similarity with $c_L^{\mathfrak{O}(X)}$ and $\overline{L}^{\mathfrak{O}(X)}$; in case of a space X where $\mathfrak{O}(X \times X)$ is naturally isomorphic to $\mathfrak{O}(X) \oplus \mathfrak{O}(X)$ it is indeed the same.)

3.3. PROPOSITION. *Let L be a Hausdorff subframe of M. Then*

$$\mathcal{E}^M(L) \subseteq \overline{L}^M.$$

If $M = \mathfrak{O}(X)$, we have, furthermore,

$$\overline{L}^M \subseteq \mathcal{F}^X(L).$$

PROOF: I. If b and $b \wedge x$ are in $\overline{L}^M$ we have

$$(*) \quad (x \oplus b) \vee c_L^M = ((x \wedge b) \oplus 1) \vee c_L^M = (1 \oplus (x \wedge b)) \vee c_L^M = (b \oplus x) \vee c_L^M.$$

Now let a, $a \vee x$ and $a \wedge x$ be in $\overline{L}^M$. Using $(*)$ repeatedly we obtain

$$\begin{aligned}(x \oplus 1) \vee c_L^M &= ((x \wedge (a \vee x)) \oplus 1) \vee c_L^M = \\ &= (x \oplus (a \vee x)) \vee c_L^M = ((x \oplus a) \vee (x \oplus x)) \vee c_L^M = \\ &= ((a \oplus x) \vee (x \oplus x)) \vee c_L^M = ((a \vee x) \oplus x) \vee c_L^M = \\ &= (1 \oplus (x \wedge (a \vee x))) \vee c_L^M = (1 \oplus x) \vee c_L^M\end{aligned}$$

so that also $x \in \overline{L}^M$. As we have, in the Hausdorff case, $L \subseteq \overline{L}^M$, we see that $\mathcal{E}^M(L) \subseteq \overline{L}^M$.

II. Now let $M = \mathfrak{O}(X)$, let $\pi_i : X \times X \to X$ be the product projections and let $\pi : M \oplus M \to \mathfrak{O}(X \times X)$ be defined by $\pi \cdot \iota_i = \mathfrak{O}(\pi_i)$. We have

$$\pi(U \oplus X) = \pi(\iota_1(U) \cap \iota_2(V)) = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times V.$$

Thus,

$$C_L^X = \pi(c_L^M).$$

Let $W \in \overline{L}^M$. We have $(W \oplus X) \vee c_L^M = (X \oplus W) \vee c_L^M$ and hence, applying π we obtain $(W \times X) \cup C_L^X = (X \times W) \cup C_L^X$. ☐

3.4. Lemma. *Let $p : X \to Y$ be a continuous mapping onto a Hausdorff Y. Let B be a subset of Y. Then*

$$(p^{-1}(B) \times X) \cup C_p^X = (X \times p^{-1}(B)) \cup C_p^X.$$

On the other hand, if A is a subset of X and if

$$(A \times X) \cup C_p^X = (X \times A) \cup C_p^X$$

then $A = p^{-1}(B)$ for a unique $B \subseteq Y$.

Proof: I. If $p^{-1}(B) = \emptyset$ or X the equality is trivial. Else let $x \notin p^{-1}(B) \ni a$. Then $p(a) \neq p(x)$ and we have disjoint open U, V separating $p(a)$ and $p(x)$. Thus, $(a, x) \in p^{-1}(U) \times p^{-1}(V) \subseteq C_p^X$.

II. Let the second statement fail. Then there is an A with $A \neq p^{-1}(p[A])$. Choose $b \in p^{-1}(p[A]) \smallsetminus A$. Then $p(b) = p(a)$ for some $a \in A$. As $(a, b) \in A \times X$ but $(a, b) \notin X \times A$, we have $(a, b) \in p^{-1}(U) \times p^{-1}(V)$ for some open disjoint U, V. But then $p(a) = p(b) \in U \cap V$. $\square$

3.5. Theorem. *Let $p : X \to Y$ be continuous onto and let Y be Hausdorff. Put $h = \mathfrak{O}(p)$, $L = \mathfrak{O}(Y)$. Then p is a quotient map iff $\mathcal{F}^X(h[L]) \subseteq h[L]$.*

Proof: Let p be a quotient and let $W \in \mathcal{F}^X(h[L])$. Then $(W \times X) \cup C_L^X = (X \times W) \cup C_L^X$ and hence, by 3.4, $W = p^{-1}(B)$ for a $B \subseteq Y$. As W is open, B is open and $W \in h[L]$. On the other hand let the equality hold and let $p^{-1}(B) = W$ be open. By 3.4, $(p^{-1}(B) \times X) \cup C_p^X = (X \times p^{-1}(B)) \cup C_p^X$ and, by the equality assumed, $p^{-1}(B) = p^{-1}(U)$ for an open U. As p is onto, $B = U$. $\square$

3.6. Theorem. *Let a T_D-space X be such that $\mathfrak{O}(X \times X) \cong \mathfrak{O}(X) \oplus \mathfrak{O}(X)$ by the natural isomorphism (for instance, locally compact). Let $\mathfrak{O}(Y)$ be Hausdorff. Then a continuous $p : X \to Y$ is a quotient iff $\mathfrak{O}(p)$ is a regular monomorphism.*

Proof: As $\mathfrak{O}(X \times X) \cong \mathfrak{O}(X) \oplus \mathfrak{O}(X)$, C_p^X coincides with the corresponding c_L^M and we have $\mathcal{F}^X(h[L]) = \overline{h[L]}^M$. Thus, the statement follows by combining 3.5 with 2.3 (and taking into account 1.7.1). $\square$

More generally, for any set $\mathcal{C}$ of complemented sublocales of X, let $f_{\mathcal{C}} : X_{\mathcal{C}} \to X$ be universal among all maps for which the inverse images of sublocales in $\mathcal{C}$ are closed. $f_{\mathcal{C}}$ is monic and epic; it can be constructed as the inverse limit of all maps $X_{\{C\}} \to X$ for $C \in \mathcal{C}$.

Aside: If $\mathcal{C}$ is the set of all closed sublocales then $X_{\mathcal{C}}$ is called the dissolution X_d, or the splitting X' of X. Its frame is the opposite of the lattice of sublocales.

Here we are interested in the case where $\mathcal{C}$ is the set $\mathcal{SC}(X)$ of scattered sublocales of X. Write $s_X : X_{sd} \to X$ for $f_{\mathcal{SC}(X)} : X_{\mathcal{SC}(X)} \to X$. So X_{sd} is the result of forcing all scattered subspaces to be discrete.

Proposition. *Let X be a strongly Hausdorff space. Then $\mathbf{pl}(X) \cong \mathbf{pl}(X_{sd})$, and X_{sd} is spatial and strongly Hausdorff.*

Proof: Consider the pullback square below.

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "s_{\mathbf{pl}(X)}"'] & X_{sd} \arrow[d, "s_X"] \\
\mathbf{pl}(X) \arrow[r] & X
\end{tikzcd}$$

Because $\mathbf{pl}(X) = \bigwedge\{X \smallsetminus S \mid S \in \mathcal{SC}(X)\}$ and pullbacks preserve meets, $P = \bigwedge\{X \smallsetminus s_X^{-1}(S) \mid S \in \mathcal{SC}(X)\}$. Since each $s_X^{-1}(S)$ is discrete (all sublocales of S are scattered) $P \geq \mathbf{pl}(X_{sd})$.

Since $\mathbf{pt}$ preserves limits and X_{sd} is the limit of maps whose underlying set maps are identities, X and X_{sd} have the same points. Each point is contained in some scattered subspace of X, P is pointless, and hence $P = \mathbf{pl}(X_{sd})$.

$s_{\mathbf{pl}(X)}$ is an isomorphism because it is universal among all maps for which the inverse images of all sublocales of the form $\mathbf{pl}(X) \wedge S$ for $S \in$ (X) are closed. But each $\mathbf{pl}(X) \wedge S = 0$. That X_{sd} inherits strongly-Hausdorffness from X follows because s_X is monic and diagonals are stable pullback along monomorphisms.

It remains to show that X_{sd} is spatial. First note that the topology on X generated by $\mathfrak{O}(X)$ and by complements of elements of $\mathcal{SC}(X)$ is the set $\mathcal{T} = \{U \smallsetminus S \mid U \in \mathfrak{O}(X) \ \& \ S \in \mathcal{SC}(X)\}$. That $\mathcal{T}$ is closed under finite meets is clear. That $\mathcal{T}$ is also closed under arbitrary unions follows from the fact that scatteredness is a local property and inherited by subspaces. ($\mathcal{T}$ is the topology which is induced by X_{sd}

surjections are stable under pullback along locally closed inclusions and all points of Y are locally closed. This in turn implies that $\mathrm{int}_Z(\zeta) = \mathbf{pt}(\zeta)$ because f is a quotient map and in **Top** $f^{-1}(\mathbf{pt}(\zeta)) = \mathbf{pt}(\eta)$. So putting these pieces together we get that

$$\begin{aligned}(fg)^{-1}(\mathrm{int}_Z(\zeta)) = (fg)^{-1}(\mathbf{pt}(\zeta)) \geq g^{-1}(\mathbf{pt}(\eta)) = \\ = g^{-1}(\mathrm{int}_Y(\eta)) = g^{-1}(\eta) = (fg)^{-1}(\zeta).\end{aligned}$$

Since the other inequality is clear it follows that fg satisfies (RE1).

To show that fg satisfies (RE2) consider an open sublocale U of X equipped with descent data relative to fg. Then U is also equipped with descent data relative to g (the kernel pair of g factors through the kernel pair of fg) and hence descends to an open sublocale V of Y. It remains to show that there exists a subspace W of Z whose inverse image under f (in **Top**) is V. This suffices, because if such a W exists it is necessarily open (f is a quotient map) and hence the inverse images of W under f in **Loc** and in **Top** coincide. Put $W = f(V)$. To show that $V = f^{-1}(W) = f^{-1}f(V)$ we have to show that for any two points y, y' which lie in the same fibre of f, $y \in V$ implies $y' \in V$. Since U is equipped with d.d. relative to fg, so is $U \wedge (fg)^{-1}(S)$ for any sublocale S of Z. Furthermore, $U \wedge (fg)^{-1}(S)$ is also equipped with d.d. relative to the restriction of fg to S. (This follows essentially because pullbacks commute with pullbacks.) Applying the preceding remarks to the sublocale $\{f(y)\}$ we get that $U \wedge (fg)^{-1}(\{f(y)\})$ is equipped with d.d. relative to the unique map $(fg)^{-1}(\{f(y)\}) \to 1$. But this means that it has to be either all of $(fg)^{-1}(\{f(y)\})$ or 0. This in turn implies that V either contains or is disjoint from $f^{-1}(\{y\})$ because the induced map $(fg)^{-1}(\{y\}) \to f^{-1}(\{y\})$ is, as the pullback of a surjection along a locally closed incluson (Z is a T_D-space), a surjection. ☐

4.5. Let C be a subset of a space X. Define a space X_C on the same set as X endowing it with the least topology finer than $\mathfrak{O}(X)$ in which C is closed, and let $j_C : X_C \to X$ be the continuous map whose underlying set map is the identity map. Similarly, we can define for each complemented sublocale C of a locale X a map $j_C : X_C \to X$ ($X_C = C \coprod (X \smallsetminus C)$). In this case j_C is universal among all continuous maps of locales into X for which the inverse image of C is closed.

3.7. Theorem. *Let $p : X \to Y$ be a continuous mapping onto, let Y satisfy (ApC). Set $L = \mathfrak{O}(Y)$, $M = \mathfrak{O}(X)$ and $h = \mathfrak{O}(p)$. Let L be Hausdorff. Then the following statements are equivalent :*

(i) *$p : X \to Y$ is a quotient mapping,*
(ii) *$h : L \to M$ is a regular monomorphism,*
(iii) *$\mathcal{F}^M(h[L]) \subseteq h[L]$,*
(iv) *$\overline{h[L]}^M = h[L]$,*
(v) *$\mathcal{E}^M(h[L]) = h[L]$.*

Proof: (i)⇒(iii) by 3.5, (iii)⇒(iv) by the second statement in 3.3, (iv)⇒(v) by the first one, and (v)⇒(i) by the Theorem quoted in 3.1. Finally, (ii)⇔(iv) by Theorem 2.3. ☐

Note : In particular the statements are equivalent whenever X and Y are metrizable.

4. Regular epimorphisms between spatial Hausdorff locales

4.1. A fact which sharply distinguishes **Loc** and **Top** is the fact that te class of regular epimorphisms in **Loc** is not closed under composition. (In **Top** regular epimorphisms are just quotient maps and these are clearly closed under composition.) Related to this is the fact that regular epimorphisms between spatial locales are not necessarily quotient maps as maps between sober spaces ([**20**]). In this section we show that even if we restrict ourselves to maps between spatial Hausdorff locales, regular epimorphism still fail to compose, and consequently that regular epimorphism between spatial Hausdorff locales are not necessarily quotient maps considered as maps between strongly Hausdorff spaces. Since regular epimorphisms between spatial metrizable locales are quotient maps ([**22**]) an interesting open problem is the problem of finding the weekest separation axiom for which regular epimorphisms and quotient maps coincide.

As we have already mentioned in the introduction we will switch to the localic perspective in this section because it is more natural whenever spaces play a major role. In particular, we will speak of regular *epimorphisms* in **Loc** instead of regular monomorphisms in **Frm** (which was more natural when dealing with various closures of subframes). By a *surjection* of locales we mean an epimorphism in **Loc**, i.e. a map for which the corresponding homomorphism in **Frm** is one-to-one. An open sublocale of a space X will be denoted simply as, say, U instead of $\hat{U}$ (recall 1.2), and the corresponding closed sublocale $\check{U}$ will be denoted by $X \smallsetminus U$.

To describe regular epimorphisms it will be convenient to use the language of descent theory. Recall that an open sublocale U of a locale A is equipped with descent data (d.d.) relative to a map $f : A \to B$ iff it has isomorphic pullbacks along the kernel pair

$$\begin{tikzcd} A \times_B A \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & A \end{tikzcd}$$

of f. (Readers unfamiliar with descent theory may take this as the definition.) If f is a regular epimorphism then U is equipped with d.d. relative to f iff there exists a unique open sublocale V of B such that $f^{-1}(V) = U$. This is just the translation into localic terms of the statement that f^* is the equalizer of π_1^* and π_2^*.

Finally, we will assume that throughout this section

all spaces are sober and satisfy the separation axiom T_D.

The latter assumption ensures that localic surjections between the spatial locales are onto on points, and also that the distinct subspaces induce distinct sublocales.

4.2. Let us recall a characterization of regular epimorphisms from [**20**]: A surjection $f : X \to Y$ of locales is said to satisfy (RE1) if

> *for each sublocale γ of Y, $f^{-1}(\gamma)$ is open only if $f^{-1}(\gamma) = f^{-1}(\mathrm{int}_Y \gamma)$,*

and to satisfy (RE2) if

> *for all open sublocales U of X equipped with d.d. relative to f there exists a sublocale γ such that $U = f^{-1}(\gamma)$.*

THEOREM. *A surjection $f : X \to Y$ of locales is a regular epimorphism if and only if it satisfies (RE1) and (RE2).*

4.3. From this theorem one infers

LEMMA. *Let X be a scattered space. Then a surjection $f : X \to Y$ is a quotient map if and only if it is a regular monomorphism in* **Loc**.

PROOF: For any sublocale γ of Y we have $f^{-1}(\gamma) = f^{-1}(\mathsf{pt}\,\gamma)$ because each sublocale of X is a space (recall(1.2.1)). So (RE1) reduces to the usual definition of quotient maps in **Top**, while (RE2) is satisfied even for arbitrary sublocales (subspaces) S of X because f is onto. $\square$

4.4. Recall that each space Y is the quotient of a scattered space: For $y \in Y$ define a space Y_y on the underlying set of Y with the same neighbourhoods of y as in Y, and all the other points isolated. Let $g_y : Y_y \to Y$ be the identity carried maps. Consider the sum

$$\iota_y : Y_y \to X = \coprod \{Y_y \mid y \in Y\}.$$

X is scattered and the map $g : X \to Y$ defined by $g\iota_y = g_y$ is clearly a quotient map

THEOREM. *Let $f : Y \to Z$ be a regular epimorphism of spatial locales. Then the following are equivalent*

(i) *f is a quotient map;*
(ii) *there exist a scattered space X and a regular epimorphism $g : X \to Y$ such that fg is again regular;*
(iii) *for all locales X and all regular epimorphisms $g : X \to Y$, $fg : X \to Z$ is again regular.*

PROOF: That (iii)⇒(ii) follows from the fact that each space is the quotient of a scattered space.

To see that (ii)⇒(i) let $g : X \to Y$ be a regular epimorphism from a scattered space onto Y such that fg is again regular. By 4.3 fg is a quotient map. But since quotient maps are closed under right cancellation f has to be a quotient map as well.

It remains to show that (i)⇒(iii). So let $f : Y \to Z$ be a quotient map, and $g : X \to Y$ an arbitrary regular epimorphism. We will show that fg satisfies (RE1) and (RE2).To show that (RE1) holds consider an arbitrary sublocale ζ of Z for which $(fg)^{-1}(\zeta)$ is open. Put $\eta = f^{-1}(\zeta)$. Then $g^{-1}(\eta)$ is open and hence, since g satisfies (RE1), $g^{-1}(\eta) = g^{-1}(\mathrm{int}_Y(\eta))$. This implies that $\mathrm{int}_Y(\eta) = \mathsf{pt}(\eta)$ because