Note on the maximum size of a Sidon set

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Abstract

We reprove an old result of Erdős and Turán which estimates from above the maximum size of a Sidon subset of $\{1, 2, \ldots, n\}$. We show a connection with counting a special kind of up-down permutations.

$\mathbf{N}$ denotes the set of positive integers $\{1, 2, \ldots\}$. A set $X \subset \mathbf{N}$ is called a *Sidon set* if $x - y = n$ has at most one solution $x, y \in X$ for any $n \in \mathbf{N}$. Let $F(n)$ be the maximum size of a Sidon subset $X \subset \{1, 2, \ldots, n\}$. The estimate

$$n^{1/2} - c_1 n^{5/16} < F(n) < n^{1/2} + c_2 n^{1/4}, \tag{1}$$

$c_1, c_2 > 0$ are constants, is a classical result. The upper bound was proved by Erdős and Turán [2] (this paper is reprinted in Turán [7]). The lower bound is due independently to Chowla [1] and Erdős [3]. For further information we refer the reader to Halberstam and Roth [5]. A new proof of the upper bound in (1) follows.

Proof. Let $1 \le x_1 < x_2 < \cdots < x_r \le n$ be a Sidon set and let $k \in \mathbf{N}, k < r$. We consider a sum S and estimate it from above and below,

$$\binom{kr - \binom{k+1}{2} + 1}{2} \le S = \sum_{0 < j-i \le k} x_j - x_i < \binom{k+1}{2} n. \tag{2}$$

To obtain the upper estimate write $x_j - x_i = (x_{i+1} - x_i) + (x_{i+2} - x_{i+1}) + \cdots + (x_j - x_{j-1})$ and observe that each term $x_l - x_{l-1}$ appears then in S at most $1 + 2 + \cdots + k$ times, that is $\binom{k+1}{2}$ times ("Gauss' lemma"). The lower estimate is immediate after realizing that S consists of $P = (r-1) + (r-2) + \cdots + (r-k)$ summands which are all *distinct* elements of $\mathbf{N}$, thus $S \geq 1 + 2 + \cdots + P$ (now we have applied "Gauss' lemma" twice).

Easy simplifications turn (2) in the inequality

$$r < \sqrt{(1+1/k)n} + (k+1)/2.$$

We can assume that $r > \lceil n^{1/4} \rceil$. Setting $k = \lceil n^{1/4} \rceil$ we finish the proof,

$$r < n^{1/2} + n^{1/4} + O(1). \tag{3}$$

□

The original proof of Erdős and Turán — see [2], [5], or [7] — is simple and elegant. However, we dare to say that the above proof is even simpler and more straightforward. For instance, in the last step of the proof of Erdős and Turán one has to choose the best m to minimize

$$\frac{n}{m} + \left(n + m + \frac{n^2}{m^2}\right)^{1/2}.$$

The optimum choice $m = \sqrt{2}n^{3/4}$ implies

$$r < n^{1/2} + \sqrt{2}n^{1/4} + O(1),$$

i.e. a bound slightly worse compared to (3).

P. Erdős conjectured [6] that in (1) $c_1 n^{5/16}$ can be replaced by c_1 and $c_2 n^{1/4}$ by $n^{o(1)}$. We believe the full potential of our approach has not yet been exploited but so far we could not obtain in (3) even the modest improvement $o(n^{1/4})$.

Let us return to the lower estimate in (2). For an s-tuple

$$Y = (y_1, y_2, \ldots, y_s) \in \mathbf{N}^s$$

and an interval $I = (y_i, y_{i+1}, \ldots, y_j), 1 \leq i \leq j \leq s$, we define $|I| = j - i + 1$ and $s(I) = \sum_{l=i}^{j} y_l$. We are interested in the quantity

$$S(s,k) = \min_Y \sum_{|I| \leq k} s(I),$$

where the minimum is taken over all s-tuples Y which have all $s(I)$ distinct. With $P(s,k)$ denoting the number of summands $s(I)$, i.e. $P(s,k)=s+(s-1)+\cdots+(s-k+1)=k(s-(k-1)/2)$, we have the trivial bound

$$S(s,k)\geq 1+2+\cdots+P(s,k)=\binom{P(s,k)+1}{2}.$$

We have used it in (2), we had $s=r-1, y_i=x_{i+1}-x_i$, and later $k\sim s^{1/2}$. An improvement here would imply an improvement in (3).

Problem 1. Improve for $k\sim s^{1/2}$ the trivial lower bound on $S(s,k)$.

It seems reasonable to consider a related quantity

$$T(s,k)=\min_{Y}\sum_{|I|\leq k}s(I),$$

now we minimize over a larger family of Y's which have totally distinct only the $s(I)$'s with $|I|\leq k$. For instance,

$$T(s,1)=1+2+\cdots+P(s,1)=\binom{s+1}{2}$$

and any permutation $Y=(y_1,\ldots,y_s)$ of $\{1,2,\ldots,s\}$ attains the minimum.

Our second group of problems arises from the natural question if $T(s,2)=1+2+\cdots+P(s,2)$. This to be true means there are natural numbers $y_1,\ldots,y_s$ so that

$$y_1, y_1+y_2, y_2, y_2+y_3, y_3, \ldots, y_{s-1}, y_{s-1}+y_s, y_s \tag{4}$$

is a permutation of $\{1,2,\ldots,2s-1\}$.

Permutations $a_1a_2\ldots a_n$ of $\{1,2,\ldots,n\}$ satisfying $a_1<a_2>a_3<\cdots$ are called *up-down* permutations, their counting is one of the little gems of enumerative combinatorics (see Graham et al. [4], p. 377). The permutation in (4) is not only up-down, it meets a much more restrictive condition — each top number is a sum of its two neighbours.

Definition 2. An up-down permutation $a_1 a_2 \ldots a_n$ of $\{1, 2, \ldots, n\}$, n is odd, is said to be a *sum-closed* permutation if $a_{2i} = a_{2i-1} + a_{2i+1}$ for $i = 1, 2, \ldots, (n-1)/2$.

For instance, for $n = 9$ the sum-closed permutations are

1 4 3 9 6 8 2 7 5   1 5 4 7 3 9 6 8 2   5 9 4 6 2 3 1 8 7
5 7 2 8 6 9 3 4 1   2 8 6 9 3 7 4 5 1   7 8 1 3 2 6 4 9 5.

Let $Y(n)$, $n \in \mathbf{N}$ is odd, be the number of sum-closed permutations of $\{1, 2, \ldots, n\}$. Reverting a sum-closed permutation produces one, so $Y(n)$ is even for $n > 1$. By hand and by computer, we have found the following values of $Y(n)$.

n	1	3	5	7	9	11	13	15	17
$Y(n)$	1	2	4	4	6	42	126	396	1786

n	19	21	23	25
$Y(n)$	7748	37962	213910	1228114

Apparently $Y(2m-1) > 0$ for any $m \in \mathbf{N}$ but it is not clear how to prove it rigorously.

Problem 3. Show there is a sum-closed permutation for any odd n.

The enumeration of sum-closed permutations is a challenging task, they do not seem to be easily tractable by the standard techniques.

Problem 4. Is it true that the numbers $Y(2m-1)$ grow exponentially, i.e.

$$Y(2m-1) \sim c^m$$

for some constant $c > 1$ and $m \to \infty$?

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