Ramsey–Type Results for Geometric Graphs. II

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Abstract

We show that for any 2–coloring of the $\binom{n}{2}$ segments determined by n points in the plane, one of the color classes contains non-crossing cycles of lengths $3, 4, \ldots, \lfloor\sqrt{n/2}\rfloor$. This result is tight up to a multiplicative constant. Under the same assumptions, we also prove that there is a non-crossing path of length $\Omega(n^{2/3})$, all of whose edges are of the same color. In the special case when the n points are in convex position, we find longer monochromatic non-crossing paths, of length $\lceil\frac{n+1}{2}\rceil$. This bound cannot be improved. We also discuss some related problems and generalizations. In particular, we give sharp estimates for the largest number of disjoint monochromatic triangles that can always be selected from our segments.

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1 Introduction

A *geometric graph* is a graph drawn in the plane so that every vertex corresponds to a point, and every edge is a closed straight-line segment connecting two vertices but not passing through a third. The $\binom{n}{2}$ segments determined by n points in the plane, no three of which are collinear, form a *complete* geometric graph with n vertices (see [PA95]). In classical Ramsey theory, we want to find large monochromatic subgraphs in a complete graph whose edges are colored with several colors [B74], [GRS90]. Most questions of this type can be formulated for complete geometric graphs, where the monochromatic subgraphs are required to satisfy certain geometric conditions. The investigation of these problems was initiated in [KPT96].

A subgraph of a geometric graph is said to be *non-crossing*, if no two of its edges have an interior point in common. In the present paper, we show how to find large non-crossing monochromatic paths and cycles in a geometric graph whose edges are colored with two colors.

Theorem 1.1 *For any 2–coloring of the edges of a complete geometric graph with n vertices, there exist non-crossing cycles of length $3, 4, \ldots, \lfloor\sqrt{n/2}\rfloor$, all of whose edges are of the same color. The order of magnitude of this bound cannot be improved.*

Theorem 1.2 *For any 2–coloring of the edges of a complete geometric graph with n vertices, there exists a non-crossing path of length $\Omega(n^{2/3})$, all of whose edges are of the same color.*

The last result improves the bound $\Omega(n^{1/2})$ established in [KPT96]. It is very likely that the $\Omega(n^{2/3})$ bound in Theorem 1.2 can be further improved to (nearly) linear. We can verify this conjecture for *convex geometric graphs*, i.e., for geometric graphs whose n points form the vertex set of a convex polygon.

Theorem 1.3 *For any 2–coloring of the edges of a complete convex geometric graph with $n \geq 3$ vertices, there exists a non-crossing monochromatic path of length $\lfloor\frac{n+1}{2}\rfloor$. This bound cannot be improved.*

The proof of Theorem 1.3 uses the following unpublished result of Micha Perles, whose proof is included.

Theorem 1.4 (Perles) *If a convex geometric graph of $n \geq k+1$ vertices has more than $\lfloor(k-1)n/2\rfloor$ edges than it contains a non-crossing path of length k. This bound cannot be improved.*

A graph is called a *caterpillar*, if it is a tree containing no three edge disjoint paths of length two starting at the same vertex. In fact, Perles proved that under the conditions of Theorem 1.4, one can always find a non-crossing subgraph isomorphic to any given caterpillar of $k+1$ vertices.

The proofs of Theorems 1.1-1.4 can be turned into $O(n^2)$ time algorithms to find monochromatic cycles resp. paths with the required properties.

The above results can be rephrased using the following notation. Let $\mathcal{G}$ be a class of (so-called *forbidden*) geometric subgraphs. We want to determine the smallest number $R = R(\mathcal{G})$ with the property that every complete geometric graph with R vertices, whose edges are colored with 2 colors, contains a monochromatic subgraph belonging to $\mathcal{G}$. If we restrict our attention to *convex* geometric graphs, then the corresponding function is denoted by $R_c(\mathcal{G})$. Clearly, $R_c(\mathcal{G}) \leq R(\mathcal{G})$.

For any positive integer k, let $k\mathcal{G}$ denote the class of all geometric graphs that can be obtained by taking the union of k pairwise disjoint members of $\mathcal{G}$, any two of which can be separated by a straight line.

Theorem 1.5 *Let $\mathcal{G}$ be any class of geometric graphs, each of which has at least two vertices.*

(i) *If k is a power of 2 then*

$$R(k\mathcal{G}) \leq (R(\mathcal{G})+1)k - 1.$$

(ii) *For any $k > 0$*

$$R(k\mathcal{G}) \leq \left\lceil \frac{3(R(\mathcal{G})+1)}{2} \right\rceil k - \left\lceil \frac{R(\mathcal{G})+1}{2} \right\rceil.$$

(iii) *For any $k > 0$*

$$R_c(k\mathcal{G}) \leq (R_c(\mathcal{G})+1)k - 1.$$

In particular, if $\mathcal{G} = \mathcal{T}$ is the class of triangles, we have $R(\mathcal{T}) = 6$. Thus, by Theorem 1.5 (i), we obtain

$$R(k\mathcal{T}) \leq 7k - 1,$$

provided that k is a power of 2. The following theorem shows that this result cannot be improved.

Theorem 1.6 *Let $\mathcal{T}$ denote the class of triangles and let k be a positive integer. Then*

$$R_\Delta(k) := R(k\mathcal{T}) \geq (R(\mathcal{T})+1)k - 1 = 7k - 1\ .$$

Using the approach of [KPTT97], one can design an $O(n^{\log\log n+2})$-time algorithm to find k pairwise *non-crossing* triangles of the same color in any complete geometric graph of $n = 7k - 1$ vertices, whose edges are colored with two colors, provided that k is a power of 2. (Note that in an abstract graph, one can always find $\lfloor n/5 \rfloor$ pairwise *vertex-disjoint* triangles of the same color [BES75].)

The paper is organized as follows. Sections 2 and 3 contain the proofs of Theorems 1.1 and 1.2, respectively. Theorems 1.3 and 1.4 are proved in Section 4, while Sections 5 and 6 contain the proofs of Theorems 1.5 and 1.6, resp.

2 Cycles – Proof of Theorem 1.1

Theorem 1.1 is an immediate consequence of the following result.

Theorem 2.1 *Let $\mathcal{C}_k$ be the class of all non-crossing cycles of length k, and let let $\mathcal{D}_k$ be the class of all non-crossing cycles of length k, triangulated from a vertex. Then*

$$\Omega(k^2) = R_c(\mathcal{C}_k) \le R(\mathcal{C}_k) \le R(\mathcal{D}_k) = O(k^2).$$

Proof. First we prove that $R_c(\mathcal{C}_k) > (k-1)^2$.

Take $(k-1)^2$ points on a circle and partition them into $k-1$ groups, each containing $k-1$ consecutive points. Color with red all edges between points in different groups, and color with blue all edges between points belonging to the same group.

Any red non-crossing cycle contains at most one point from each group, hence it can not have more than $k-1$ points. On the other hand, all vertices of a blue cycle are from the same group, so there is no blue cycle with more than $k-1$ points.

Next we show that $R(\mathcal{D}_k) \le 2(k-1)(k-2) + 2$.

Let P denote the vertex set of a complete geometric graph G of $2(k-1)(k-2)+2$ vertices, whose edges are colored with red and blue. Let p be a vertex of the convex hull of P. Since there are $2(k-1)(k-2)+1$ edges incident to p, at least $(k-1)(k-2)+1$ of them are of the same color, say, red. Let $p_1, p_2, \ldots, p_{(k-1)(k-2)+1}$ be vertices of G, listed in clockwise order of visibility from p, such that each edge pp_i is red. A path $p_{i_1}p_{i_2}\ldots p_{i_j}$ is said to be *monotone* if $i_1 < i_2 < \ldots < i_j$.

Define a partial ordering of the vertices $p_1, p_2, \ldots, p_{(k-1)(k-2)+1}$, as follows. Let $p_i < p_j$ if $i < j$ and there is a monotone red path connecting p_i to p_j. By Dilworth's theorem [D50], there are either $k-1$ elements that form a totally ordered subset, or k elements that are pairwise incomparable. In the first case, there is a monotone red path $q_1, q_2, \ldots, q_{k-1}$ and we can complete it to a non-crossing red cycle $p, q_1, q_2, \ldots, q_{k-1}$ of size k, together with the corresponding diagonals from p. In the second case, there is a complete blue subgraph of k vertices $q_1, q_2, \ldots, q_k$ because any two incomparable elements are connected by a blue edge. By a result of Gritzmann et al. [GMPP91, PA95], this contains a non-crossing cycle of length k, triangulated from a vertex. $\square$

3 Paths – Proof of Theorem 1.2

Let G be a complete geometric graph whose edges are colored red and blue. Suppose that G contains no monochromatic non-crossing path of length m. To establish Theorem 1.2, it is enough to prove that G has at most $O(m^{3/2})$ vertices.

Suppose without loss of generality that no two vertices lie on a vertical line. For each vertex v of G, let $b(v)$ (resp. $r(v)$) denote the length of the longest x-monotone blue path (resp. red) in G whose rightmost vertex is v.

Since every monotone path is non-crossing, $b(v)$ and $r(v)$ are integers between 0 and $m-1$. Observe that $(b(v), r(v)) \neq (b(v'), r(v'))$ for $v \neq v'$. Indeed, if $v \neq v'$ are two vertices of G such that v' lies to the right of v, then $b(v') > b(v)$ (if vv' is colored blue) or $r(v') > r(v)$ (if vv' is colored red). As a consequence, G has at most m^2 vertices. The following more careful analysis gives a better bound.

Define $\mathcal{U}$ as the set of pairs with $(b, r) = (b(u), r(u))$ for some vertex u of G. For $0 \leq b_1 \leq b_2 \leq m$ and $0 \leq r_1 \leq r_2 \leq m$, define a subset $\mathrm{Rect}(b_1, b_2, r_1, r_2)$ of $\mathcal{U}$ by

$$\mathrm{Rect}(b_1, b_2, r_1, r_2) = \{(b, r) \in \mathcal{U} \mid b_1 \leq b < b_2, r_1 \leq r < r_2\}.$$

Every pair (b, r), $0 \leq b \leq m$, $0 \leq r \leq m$, defines a partition of $\mathcal{U}$ into the following four subsets:

$$\begin{aligned} Q^{++}(b,r) &= \mathrm{Rect}(b, m, r, m), & Q^{+-}(b,r) &= \mathrm{Rect}(b, m, 0, r), \\ Q^{-+}(b,r) &= \mathrm{Rect}(0, b, r, m), & Q^{--}(b,r) &= \mathrm{Rect}(0, b, 0, r). \end{aligned}$$

The key observation is the following.

Lemma 3.1 *For any $t \in \{1, 2, \ldots, 2m-1\}$, there are two integers $b(t), r(t) \in \{0, 1, \ldots, m\}$ such that $b(t) + r(t) = t$ and*

$$|Q^{+-}(b(t), r(t)) \cup Q^{-+}(b(t), r(t))| \leq 3m.$$

Before giving the proof of Lemma 3.1, we finish the proof of Theorem 1.2. Obviously, it suffices to show that $|\mathcal{U}| = O(m^{3/2})$.

Suppose for simplicity that $s = \sqrt{2m}$ is an integer. Somewhat inaccurately, we shorten the notation $Q^{++}(b(t), r(t))$ to $Q^{++}(t)$. We also shorten $Q^{+-}(b(t), r(t))$, $Q^{-+}(b(t), r(t))$, and $Q^{--}(b(t), r(t))$ analogously. We have

$$\mathcal{U} = \bigcup_{t=s,2s,\dots,s^2-s} \left(Q^{+-}(t) \cup Q^{-+}(t)\right) \cup \bigcup_{t=0,s,2s,\dots,s^2-s} \left(Q^{++}(t) \cap Q^{--}(t+s)\right),$$

because each pair $(b, r) \in \mathcal{U}$ with $ks \leq b+r < (k+1)s$ lies in the first union (in a term with $t = ks$ or with $t = (k+1)s$) or in the term $Q^{++}(ks) \cap Q^{--}(ks+s)$ of the second union. Consequently,

$$|\mathcal{U}| \leq \sum_{t=s,2s,\dots,s^2-s} \left|Q^{+-}(t) \cup Q^{-+}(t)\right| + \sum_{t=0,s,2s,\dots,s^2-s} \left|Q^{++}(t) \cap Q^{--}(t+s)\right|.$$

By Lemma 3.1, each term in the first sum is at most $3m$, and it is easy to check that each term in the second sum is at most $(s/2)^2 = m/2$. It follows that

$$|\mathcal{U}| \leq (s-1) \cdot 3m + s\frac{m}{2} = O(m^{3/2}). \quad \square$$

It remains to prove Lemma 3.1.

Proof of Lemma 3.1. Suppose without loss of generality that $t \leq m$. For each $b = 0, 1, \dots, t$, denote

$$\Delta(b) = |Q^{-+}(b, t-b)| - |Q^{+-}(b, t-b)|.$$

It follows from

$$\Delta(0) \leq 0 \leq \Delta(m)$$

and from

$$|\Delta(b+1) - \Delta(b)| = |\mathrm{Rect}(b, b+1, 0, m)| + |\mathrm{Rect}(0, m, t-b-1, t-b)| \leq$$

$$\leq m + m = 2m$$

that there is a $b_0 \in \{0, \dots, m\}$ with

$$|\Delta(b_0)| \leq m.$$

We now show that the lemma holds with $b(t) = b_0$ and $r(t) = t - b_0$. Suppose this is not true, i.e.,

$$|Q^{+-}(t)| + |Q^{-+}(t)| \geq 3m + 1.$$

Then $|\Delta(b(t))| \leq m$ gives

$$|Q^{+-}(t)| \geq m+1, \qquad |Q^{-+}(t)| \geq m+1.$$

Consequently, there is a vertical line ℓ such that a set V_1 of at least $\frac{m+1}{2}$ vertices to the left of ℓ corresponds to a subset of $Q^{+-}(t)$ or $Q^{-+}(t)$, and a set V_2 of at least $\frac{m+1}{2}$ vertices to the right of ℓ corresponds to a subset of $Q^{-+}(t)$ or $Q^{+-}(t)$, respectively. All edges of the complete bipartite geometric graph $(V_1 \cup V_2, V_1 \times V_2)$ are colored by the same color: red, if V_1 corresponds to elements of $Q^{+-}(t)$, and blue, otherwise. It is well-known (e.g., see [AGHN97]) that if V_1, $|V_1| \geq \frac{m+1}{2}$, is separated by a line from V_2, $|V_2| \geq \frac{m+1}{2}$, then the graph $(V_1 \cup V_2, V_1 \times V_2)$ contains a non-crossing path of length m, – a contradiction. □

In Theorem 1.2, the edges of a complete geometric graph G are colored by two colors. If they are colored by $r \geq 2$ colors, we obtain the following.

Theorem 3.2 *Every complete geometric graph of n vertices whose edges are colored by $r \geq 2$ colors, contains a non-crossing path of length $\Omega(n^{2/(2r-1)})$, all of whose edges are of the same color.*

Proof. Let the edges of G be colored by $1, 2, \ldots, r$. Suppose that G contains no non-crossing path of length m. For each color i, define a function f_i on the vertex set of G such that $f_i(v)$ is the length of the longest x-monotone monochromatic path of color i, whose rightmost point is v. Thus, for any vertex v, we have an r-tuple $f(v) = (f_1(v), \ldots, f_r(v))$ of integers between 0 and $m-1$. Of course, $f(v) \neq f(v')$ for $v \neq v'$, since $f(v)$ and $f(v')$ differ in the i-th coordinate, where i is the color of the edge vv'. We now show by induction on r that the number of vertices of G is at most $O(m^{r-1/2})$. For $r = 2$, this was shown in the proof of Theorem 1.2. Suppose now that $r > 2$ and that the statement holds for $r-1$. Define $\mathcal{U}$ as the set of all r-tuples $f(v)$, where v is a vertex of G. The set $\mathcal{U}$ is a disjoint union of m sets $\mathcal{U}_j, j = 0, 1, \ldots, m-1$, where $\mathcal{U}_j$ consists of the r-tuples of $\mathcal{U}$ whose last coordinate is j. For each j, all edges between the vertices corresponding to elements of $\mathcal{U}_j$ are colored by colors $1, 2, \ldots, r-1$. By the inductive assumption, the size of $\mathcal{U}_j$ is at most $O(m^{r-3/2})$. Consequently, the size of $\mathcal{U}$ is at most $O(m^{r-1/2})$, and the result follows. □

4 Paths in convex geometric graphs – Proof of Theorems 1.3 and 1.4

The proof of Theorem 1.3 is based on a result of Perles (Theorem 1.4), whose proof is as follows.

Proof of Theorem 1.4. Let G be a convex geometric graph with vertices $u_1, u_2, \ldots, u_n$ in clockwise order and with $|E(G)| = m > \lfloor (k-1)n/2 \rfloor$. For convenience, let $u_0 = u_n$ and $u_{n+i} = u_i$.

Fix $E_0 = \emptyset$, and define $E_1, E_2, \ldots, E_m$ inductively, as follows. There exists an edge $e_i \in \overline{E}_{i-1} = E(G) \setminus E_{i-1}$ such that one of the closed halfplanes determined by e_i does not fully contain any element of $E(G) \setminus (E_{i-1} \cup \{e_i\})$. Such an edge will be called an *extreme* edge with respect to $\overline{E}_{i-1}$. Let $E_i = E_{i-1} \cup \{e_i\}$.

We also define some nonnegative integers $c_i(u_j)$ and $d_i(u_j)$. For any vertex u_j, let $I = I(u_j)$ be the smallest integer such that E_I contains all edges of G incident to u_j. Suppose first that $i < I$. Then, starting at u_j and visiting the vertices of the G in clockwise order, let u_p and u_q be the first and the last vertex such that $u_j u_p, u_j u_q \in \overline{E}_i$. Let $c_i(u_j)$ (resp. $d_i(u_j)$) denote the length of the longest non-crossing path in E_i starting at u_j, all of whose vertices are in $\{u_j, u_{j+1}, \ldots, u_{p-1}\}$ (resp. $\{u_{q+1}, u_{q+2}, \ldots, u_j\}$). Next, if $i = I$, then $e_i = u_j u_p$ for some vertex u_p. Since e_i is an extremal edge with respect to $\overline{E}_{i-1}$, we may suppose, by symmetry, that $\overline{E}_i$ has no element on the vertex set $\{u_j, u_{j+1}, \ldots, u_{p-1}, u_p\}$. If this is the case, then let $c_i(u_j)$ (resp. $d_i(u_j)$) denote the length of the longest non-crossing path in E_i starting at u_j, all of whose vertices lie in $\{u_j, u_{j+1}, \ldots, u_p\}$ (resp. $\{u_{p+1}, u_{p+2}, \ldots, u_j\}$). (Otherwise we change the orientation.) Finally, if $i > I$ then put $c_i(u_j) = c_I(u_j)$ and $d_i(u_j) = d_I(u_j)$.

Claim 4.1 *For every* $0 \le i \le m$,

$$\sum_{j=1}^{n} c_i(u_j) + \sum_{j=1}^{n} d_i(u_j) \ge 2|E_i|.$$

The claim is clearly true for $i = 0$. Let $0 < i \le m$, and assume that the assertion has already been proved for $i - 1$. Suppose that the endpoints of e_i are u_j and u_l. Since $c_i(u) \ge c_{i-1}(u)$ and $d_i(u) \ge d_{i-1}(u)$ for every fixed vertex u, it is enough to prove that

$$c_i(u_j)+d_i(u_j)+c_i(u_l)+d_i(u_l) \ge c_{i-1}(u_j)+d_{i-1}(u_j)+c_{i-1}(u_l)+d_{i-1}(u_l)+2.$$

However, we have either $c_i(u_j) > d_{i-1}(u_l)$ and $d_i(u_l) > c_{i-1}(u_j)$ or $c_i(u_l) > d_{i-1}(u_j)$ and $d_i(u_j) > c_{i-1}(u_l)$, depending on which side of $e_i = u_j u_l$ contains no edge of $\overline{E}_i$, and the statement follows.

Now it is easy to complete the proof of Theorem 1.4. Since $|E_m| > \lfloor (k-1)n/2 \rfloor$, there is $1 \le j \le n$ for with $c_m(u_j) + d_m(u_j) \ge k$. By the definition of c_m and d_m, this means that there is a non-crossing path of length k, passing through u_j.

The following construction shows that the bound in Theorem 1.4 is tight.

Construction. Let $0, 1, \ldots, n-1 \pmod{n}$ denote the vertices of a regular n-gon.

(i) If $n \equiv k \pmod{2}$, then connect each vertex to the $k-1$ vertices furthest from it.

(ii) If $n \not\equiv k \pmod{2}$, then connect each vertex i to the $k-2$ vertices furthest from i, and for $0 \le i \le \lfloor \frac{n}{2} \rfloor - 1$, also to $i + \frac{n-k-1}{2}$.

In case (i), every edge has at least $\frac{n-k}{2}$ vertices on both sides. In particular, this holds for the first and last edges of any non-crossing path. This implies that any such path misses at least $n-k$ vertices, so it is of length at most $k-1$.

In case (ii), suppose that $k+3 \le n$. Consider any non-crossing path P. Call an edge of P extremal if the whole path P lies in one of the closed halfspaces determined by P. If P has at least three extremal edges, then it has at most $n - 3(\frac{n-k-1}{2}) \le k$ vertices, showing that its length is at most $k-1$. However, if P has only two extremal edges, then P is a "zigzag", in particular, for each j, $\lfloor \frac{n}{2} \rfloor \ge j \ge \frac{n-k+1}{2}$ it has at most two edges of the form $(i, i+j)$. If at most one edge of P is of the form $(i, i+\frac{n-k+1}{2})$, then P contains at most $k-1$ edges. Otherwise, the first and the last edges of P are of the form $(i, i+\frac{n-k+1}{2})$ and $(j, j+\frac{n-k+1}{2})$ with $0 \le i < i + \frac{n-k+1}{2} < j \le \lfloor \frac{n}{2} \rfloor - 1$. In this case, the length of P is at most

$$2\Big(j - \Big(i + \frac{n-k+1}{2}\Big) + 1\Big) \le 2\Big(\Big\lfloor \frac{n}{2} \Big\rfloor - \frac{n-k+1}{2}\Big) \le k-1\ ,$$

as reqired.

The case $n = k+1$ can be treated similarly. $\Box$

We prove the following equivalent form of Theorem 1.3.

Theorem 4.2 *Let $\mathcal{P}_k$ be the class of all non-crossing paths of length k. Then $R_c(\mathcal{P}_k) = 2k-1$ unless $k=1$.*

Proof. First, we show that $R_c(\mathcal{P}_k) > 2k-2$. Let G be a convex geometric graph on $2k-2$ vertices, and let $p, v_1, v_2, \ldots, v_{k-2}, q, u_1, u_2, \ldots, u_{k-2}$ be its vertices listed in clockwise order. For all i, j, color all edges (v_i, v_j), (p, v_i), and (q, v_i) blue; (u_i, u_j), (p, u_i), and (q, u_i) red; (v_i, u_j) red if $i+j$ is odd and blue if $i+j$ is even. The edge (p, q) can have any color. It is not difficult to check that this graph contains no non-crossing monochromatic path of length k.

For the upper bound, first let k be even and take a convex complete geometric graph on $2k-1$ vertices, all of whose edges are colored red or blue. This graph has $(2k-1)(k-1)$ edges, so one of the color classes (say, red) contains at least $\lceil(2k-1)(k-1)/2\rceil > \lfloor(2k-1)(k-1)/2\rfloor$ edges. Therefore, by Theorem 1.4, there is a non-crossing red path of length k.

If k is odd, then we have to be more careful. Suppose there is a 2-coloring of the edges of a complete geometric graph G on $2k-1$ vertices without a non-crossing path of length k. It follows from Theorem 1.4 that $(2k-1)(k-1)/2$ edges are red and the same number of edges are blue. Let G_b denote the subgraph of G obtained from G by removing all red edges. Analyzing the proof of Theorem 1.4 for G_b, we obtain that (in the notation of the proof of Theorem 1.4)

$$\sum_{j=1}^{n} c_i(u_j) + \sum_{j=1}^{n} d_i(u_j) = 2|E_i|, \qquad i = 1, 2, \ldots, m.$$

Consequently,

$$c_i(u_j)+d_i(u_j)+c_i(u_l)+d_i(u_l) = c_{i-1}(u_j)+d_{i-1}(u_j)+c_{i-1}(u_l)+d_{i-1}(u_l)+2.$$

(Otherwise, we would have $c_m(u_j) + d_m(u_j) \geq k$ for some j, and we could find a non-crossing path of length k.) This means, for example, that if $e_i = u_j u_l$ and $\overline{E}_i$ has no element on the vertex set $\{u_j, u_{j+1}, \ldots, u_{l-1}, u_l\}$, then $c_i(u_j) = d_{i-1}(u_l) + 1$, $d_i(u_l) = c_{i-1}(u_j) + 1$.

Looking at all possible ways how we may arrange the edges of G_b in the sequence $e_1, e_2, \ldots, e_m$, we obtain the following proposition.

Proposition 4.3 *Let* $v, w \in \{1, 2, \ldots 2k-1\}$.

(a) *If* $u_v u_w$ *is blue, then at least one of the longest blue non-crossing paths starting at* u_v*, all of whose vertices lie in* $\{u_v, u_{v+1}, \ldots, u_w\}$*, contains the edge* $u_v u_w$*.*

(b) *If* $u_v u_w$ *is blue, then at least one of the longest blue non-crossing paths starting at* u_w*, all of whose vertices lie in* $\{u_v, u_{v+1}, \ldots, u_w\}$*, contains the edge* $u_v u_w$*.*

(c) *If* $u_v u_w$ *is red, then at least one of the longest blue non-crossing paths starting at* u_v*, all of whose vertices lie in* $\{u_v, u_{v+1}, \ldots, u_w\}$*, does not contain* u_w*.*

(d) *If* $u_v u_w$ *is red, then at least one of the longest blue non-crossing paths starting at* u_w*, all of whose vertices lie in* $\{u_v, u_{v+1}, \ldots, u_w\}$*, does not contain* u_v*.*

Exchanging the roles of the colors, one can formulate a similar statement about red paths in the graph G_r obtained from G by deleting all blue edges.

Consider the $2k-1$ edges u_iu_{i+1}. At least k of them are colored by the same color (blue, say). Let $u_pu_{p+1}\dots u_q$ be the longest non-crossing blue path using some of the edges u_iu_{i+1} (certainly, its length is at least 2). First, observe that Proposition 4.3 (c) yields that all edges within $\{u_p, u_{p+1}, \dots, u_q\}$ are blue. Indeed, if $u_su_t, p \le s < t \le q$, was a red edge such that all other edges $u_{s'}u_{t'}, s \le s' < t' \le t$ were blue, then Proposition 4.3 (c) would be false with $u = s$ and $w = t$. This implies that the blue path $u_pu_{p+1}\dots u_q$ cannot visit all or all but one of the vertices of G. By the maximality of this path, the edges $u_{p-1}u_p$ and u_qu_{q+1} are red. It follows from the the dual of Proposition 4.3 (a) and (c) with $v = p-1$ and $w = q+1$ that u_pu_{q+1} is blue. Analogously, $u_{p-1}u_q$ is also blue.

Suppose now that some of the edges $u_{p-1}u_r, r = p+1, p+2, \dots, q-1$, are red. Take one of them for which r is maximum. Then Proposition 4.3 (b) would be violated for $v = p-1$ and $w = r+1$. Thus, all edges $u_{p-1}u_r, r = p+1, p+2, \dots, q$, are blue. Analogously, all edges $u_{q+1}u_r, r = p, p+1, \dots, q-1$, are also blue.

The edge $u_{p-2}u_{p-1}$ cannot be red. Otherwise, starting with the longest non-crossing red path $u_pu_{p-1}\dots$ and using a similar argument as above, we would conclude that $u_{p-1}u_{p+1}$ must be red, but we already know that it is blue. Thus, $u_{p-2}u_{p-1}$ is blue, and we get a contradiction with Proposition 4.3 (a) or (c) for $v = p-2$ and $w = p+1$. □

5 General estimates – Proof of Theorem 1.5

For any set of n points P in the plane, an i-element subset of P is called an *i-set* if it can be obtained by intersecting P with an open half-plane. It is easy to see that all i-sets can be generated by the following procedure [ELSS73]: Take an oriented line ℓ passing through precisely one point $p \in P$ and having i elements of P on its left side. Rotate ℓ around p in the clockwise direction until it hits another point $q \in P$, and then continue the rotation around q, etc. Whenever ℓ passes through only one element of P, the points lying on its left side form an i-set.

Any geometric subgraph $K_{R(\mathcal{G})}$ induced by an $R(\mathcal{G})$-element subset of P contains a subgraph of the class $\mathcal{G}$, all of whose edges are of the same color. If these edges are red (blue), we say that the *type* of the corresponding subset is *red (blue)*. Note that a set may have both types.

Lemma 5.1 *Let $\mathcal{G}$ be any class of geometric graphs. Then $R(2\mathcal{G}) \le 2R(\mathcal{G}) + 1$.*

Let $N = 2R(\mathcal{G}) + 1$ and let P be the vertex set of a complete geometric graph K_N whose edges are colored by red and blue.

If all $R(\mathcal{G})$-sets of P have the same type, then take two disjoint $R(\mathcal{G})$-sets. Both contain a monochromatic geometric subgraph belonging to $\mathcal{G}$, so their union is a monochromatic member of $2\mathcal{G}$.

On the other hand, if there are two $R(\mathcal{G})$-sets of different types, then we can also find two $R(\mathcal{G})$-sets, A and B, of different types such that $|A \cup B| = R(\mathcal{G}) + 1$ and that $P \setminus (A \cup B)$ is also an $R(\mathcal{G})$-set. The type of $P \setminus (A \cup B)$ will coincide with the type of A or B. $\square$

Proof of Theorem 1.5. Part (i) is an immediate corollary of Lemma 5.1.

Let $\mathcal{G}$ be a (fixed) class of geometric graphs. Let $r = r(k, l)$ be the the smallest number with the property that every complete geometric graph with r vertices, whose edges are colored by red and blue, contains either a subgraph belonging to $k\mathcal{G}$, all of whose edges are red, or a subgraph belonging to $l\mathcal{G}$, all of whose edges are blue. So, $R = R(\mathcal{G}) = r(1, 1)$, $R(k\mathcal{G}) = r(k, k)$.

Lemma 5.2 *For* $k \geq l \geq 1$

$$r(k, l) = r(l, k) \leq (R + 1)k + \left\lceil \frac{R+1}{2} \right\rceil l - \left\lceil \frac{R+1}{2} \right\rceil.$$

Proof. By simmetry, clearly $r(k, l) = r(l, k)$. We proceed by induction on k and l. First we show that $r(k, 1) \leq kR$. Let G be a complete geometric graph with kR vertices, whose edges are colored by red and blue. Suppose that no two vertices determine a vertical line. Then we can choose $k - 1$ vertical lines that divide the vertex set of G into k equal parts. Since each part contains R of the vertices, the subgraph spanned by every part contains a monochromatic copy of a geometric graph from the class $\mathcal{G}$. Either all of them are red or one of them is blue. Therefore, the lemma holds for any pair (k, l), where either $k = 1$ or $l = 1$.

Let $k \geq l$ be fixed, and suppose that we already know that the Lemma is true for every pair $(k', l') \neq (k, l)$, where $k' \geq k, l' \geq l$.

Consider a complete geometric graph G with vertex set V, whose edges are colored by red and blue and

$$|V| = (R + 1)k + \left\lceil \frac{R+1}{2} \right\rceil l - \left\lceil \frac{R+1}{2} \right\rceil.$$

By Theorem 1.5 (i), for any $n \geq 1$, if $f(n) = 2^n(R + 1) - 1 \leq |V|$, then any geometric subgraph induced by an $f(n)$-set contains a monochromatic subgraph from the class $2^n\mathcal{G}$. Its color will be the *type* of the corresponding $f(n)$-set.

If there are two $f(n)$-sets of different types, we can cut off an $(f(n)+1)$-set which contains an $f(n)$-set of both types. Since

$$(R+1)k + \left\lceil \frac{R+1}{2} \right\rceil l - 2^n(R+1) \geq (R+1)(k-2^n) + \left\lceil \frac{R+1}{2} \right\rceil (l-2^n),$$

we can apply the induction hypothesis for the subgraph induced by the rest of the vertices, with $k' = k - 2^n, l' = l - 2^n$.

So we can assume that for all n, all $f(n)$-sets are of the same type, provided $f(n) = 2^n(R+1) - 1 \leq |V|$. Suppose that for $n_1 < n_2$, the $f(n_1)$ and $f(n_2)$-sets are of different types. Then, there exists an $n_1 \leq n < n_2$ such that the $f(n)$ and $f(n+1)$-sets are also of different types. Cut off an $f(n+1)$-set. Since it contains an $f(n)$-set, the subgraph induced by the $f(n+1)$-set contains a blue (resp. red) subgraph from the class $2^{n+1}\mathcal{G}$ and a red (resp. blue) subgraph from the class $2^n\mathcal{G}$.

If the $f(n+1)$-sets are of type blue, then either $l \leq 2^{n+1}$ and we are done, or we can apply the induction hypothesis for the graph induced by the remaining vertices with $k' = k - 2^n$ and $l' = l - 2^{n+1}$ since it is easy to check that

$$(R+1)k + \left\lceil \frac{R+1}{2} \right\rceil l - 2^{n+1}(R+1) \geq (R+1)(k-2^n) + \left\lceil \frac{R+1}{2} \right\rceil (l-2^{n+1}).$$

On the other hand, if the $f(n+1)$-sets are of type red then we can assume that $k > 2^{n+1}$ and $l > 2^n$ and we can apply the induction hypothesis with $k' = k - 2^{n+1}$ and $l' = l - 2^n$ since

$$(R+1)k + \left\lceil \frac{R+1}{2} \right\rceil l - 2^{n+1}(R+1) \geq (R+1)(k-2^{n+1}) + \left\lceil \frac{R+1}{2} \right\rceil (l-2^n)$$

and

$$(R+1)k + \left\lceil \frac{R+1}{2} \right\rceil l - 2^{n+1}(R+1) \geq (R+1)(l-2^n) + \left\lceil \frac{R+1}{2} \right\rceil (k-2^{n+1}).$$

Therefore, in the sequel we assume that for *all* n, all $f(n)$-sets are of the same type. For simplicity we assume that they are of type *red*. The case where these sets are of type *blue* can be settled very similarly. Write k in base 2, that is, $k = 2^{n_1} + 2^{n_2} + \ldots + 2^{n_a}$, $n_1 < n_2 < \ldots < n_a$. Let $V_0 = V$. By the assumption, every $f(n_1)$- and $f(n_2)$-set of V_0 are of type red. Let S_1 be an $f(n_1)$-set and let $V_1 = V_0 \setminus S_1$. If every $f(n_2)$- and $f(n_3)$-set of V_1 are of type red, then let S_2 be such an $f(n_2)$-set and let $V_2 = V_1 \setminus S_2$. In general, if V_i has already been defined by this procedure and every $f(n_{i+1})$- and $f(n_{i+2})$-set of V_i are of type red, then let S_{i+1} be any $f(n_{i+1})$-set and let $V_{i+1} = V_i \setminus S_{i+1}$. Otherwise, if not all $f(n_{i+1})$-sets or not all $f(n_{i+2})$-sets of V_i are of type red, or $i = a-1$, then stop and let $j = i$.

If $j = a-1$, then V_{a-1} still has an $f(n_a)$-set, S_a of type red, and $S_1 \cup S_2 \cup \ldots \cup S_a$ contains a red subgraph from the class $k\mathcal{G}$ and we are done. Suppose $j < a-1$. We distinguish two cases.

a. *There is an $f(n_{j+1})$-set of V_j of type blue.* Since all $f(n_{j+1})$-sets of V_{j-1} were of type red, at least one of the $f(n_{j+1})$-sets of V_j is of type red. Therefore, we can find an $(f(n_{j+1})+1)$-set, S_{j+1}, which contains an $f(n_{j+1})$-set of both types. $|S_1 \cup S_2 \cup \ldots \cup S_{j+1}| \leq (2^{n_1} + 2^{n_2} + \ldots + 2^{n_{j+1}})(R+1)$ and $S_1 \cup S_2 \cup \ldots \cup S_{j+1}$ contains a red subgraph from the class $(2^{n_1} + 2^{n_2} + \ldots + 2^{n_{j+1}})\mathcal{G}$ and a blue subgraph from the class $2^{n_{j+1}}\mathcal{G}$. If either $k \leq 2^{n_1} + 2^{n_2} + \ldots + 2^{n_{j+1}}$ or $l \leq 2^{n_{j+1}}$ then we are done. Otherwise, easy calculation shows that we can apply the induction hypothesis for for the graph induced by the remaining vertices with $k' = k - (2^{n_1} + 2^{n_2} + \ldots + 2^{n_{j+1}})$ and $l' = l - 2^{n_{j+1}}$.

b. *Every $f(n_{j+1})$-set of V_i are of type red, but there is an $f(n_{j+2})$-set of V_j of type blue.* In this case, there exists an $n_{j+1} \leq n < n_{j+2}$ such that every $f(n)$-set of V_i are of type red, but there is an $f(n+1)$-set of V_j of type blue. Then let S_{j+1} be an $f(n+1)$-set of V_j of type blue. Clearly, it contains an $f(n)$-set of V_j, which is of type red. $|S_1 \cup S_2 \cup \ldots \cup S_{j+1}| \leq (2^{n_1} + 2^{n_2} + \ldots + 2^{n_j} + 2^{n+1})(R+1)$ and $S_1 \cup S_2 \cup \ldots \cup S_{j+1}$ contains a red subgraph from the class $(2^{n_1} + 2^{n_2} + \ldots + 2^{n_j} + 2^n)\mathcal{G}$ and a blue subgraph from the class $2^{n+1}\mathcal{G}$. Again, easy calculation shows that we can proceed as in case a. □

Return to the proof of Theorem 1.5. Part (ii) is an immediate corollary of Lemma 5.2. In part (iii), if all $R_c(\mathcal{G})$-sets have the same type then, using the convexity of the geometric graph, we can take k disjoint $R_c(\mathcal{G})$-sets and we get a monochromatic geometric graph from $k\mathcal{G}$.

If there are two $R(\mathcal{G})$-sets of different types, we can proceed by induction as in part (ii). □

6 Triangles – Proof of Theorem 1.6

The proof is quite technical, therefore we sketch here only the main ideas.

It is sufficient to show that $R_\Delta(k) > 7k-7$ for every k. Indeed, if $R_\Delta(k) \leq 7k-2$ for some positive integer k, an application of Theorem 1.5 with $k = 8$ and $\mathcal{G} = k\mathcal{T}$ would yield $R_\Delta(8k) \leq 7(8k) - 9$, a contradiction.

Construction. Let k be a positive integer, $n = 7k-7$. Let $0, 1, \ldots, 7k-8$ denote the vertices of a regular n-gon, in clockwise order. If two vertices are separated by $m-1$ others, then we say that the *length* of the edge connecting them is m ($m \leq \lfloor n/2 \rfloor$). A vertex is colored red if $i \equiv 0$, 1, 3, or 5 (mod 7), and blue otherwise.

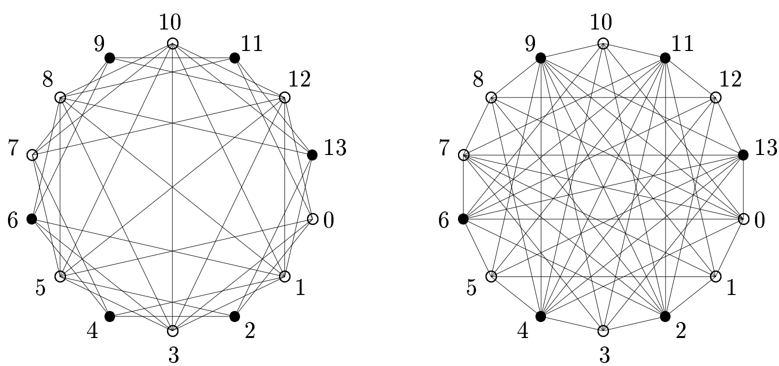
We color the edges ij ($0 \leq i \neq j \leq n-1$) according to the following rules. Let λ denote the length of ij.

(a) If $\lambda \equiv 1, 4,$ or $6 \pmod 7$, let ij be blue.

(b) If $\lambda \equiv 2$ or $3 \pmod 7$, let ij be red.

(c) If $\lambda \equiv 5 \pmod 7$, let ij be colored with the color of its starting point in the clockwise direction.

(d) If $\lambda \equiv 0 \pmod 7$, let ij be colored with the color other than the color of $i+1 \pmod 7$.



Red edges

Blue edges

Fig. 1

To see that this graph does not contain n pairwise non-crossing blue triangles, we need the following lemma that can be proved by induction. By an *arc* we mean a set of consecutive vertices.

Lemma 6.1 *Suppose that an arc I of $m \leq n/2$ vertices contains the vertices of h pairwise non-crossing blue triangles. Then $m \geq 7h-1$. Moreover, if $m = 7h-1$, then the starting point of I is blue.*

Consider a maximal set of pairwise non-crossing blue triangles. Suppose first that the points can be partitioned into three arcs, I_1, I_2, I_3, with $m_i = |I_i| \leq n/2$ such that the vertices of each of these triangles belong to the same arc. Let k_i denote the number of triangles whose vertices belong to I_i. By Lemma 6.1, we have

$$7n-7 = m_1+m_2+m_3 \geq (7k_1-1)+(7k_2-1)+(7k_3-1) > 7(k_1+k_2+k_3)-7,$$

and we can conclude that $k_1 + k_2 + k_3 < n$.

Otherwise, there is a blue triangle so that the convex hull of its vertex set contains the center of our regular n-gon. Its vertices, p, q, r, partition the remaining set of points into three arcs of lengths m_1, m_2, m_3 containing k_1, k_2, k_3 triangles, respectively. From Lemma 6.1 we obtain

$$7n-7 = m_1+m_2+m_3+3 \geq (7k_1-1)+(7k_2-1)+(7k_3-1)+3 = 7(k_1+k_2+k_3),$$

implying $n \geq k = k_1 + k_2 + k_3 + 1$. Suppose that $n = k$. Then we have equality in the previous inequalities, therefore, the length of each side of triangle pqr is divisible by 7. On the other hand, by Lemma 6.1, all of the points $p+1, q+1, r+1$ are blue, so pqr is a monochromatic red triangle, a contradiction.

The existence of n non-crossing red triangles can be excluded by a similar argument.

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