On geometric graphs with no k pairwise parallel edges*

PAVEL VALTR

Department of Applied Mathematics, Charles University
Malostranské nám. 25, 118 00 Praha 1, Czech Republic
e-mail: `valtr@kam.ms.mff.cuni.cz`

DIMACS Center, Rutgers University
P.O.Box 1179, Piscataway, NJ 08855, U.S.A.
e-mail: `valtr@dimacs.rutgers.edu`

Abstract

A *geometric graph* is a graph $G = (V, E)$ drawn in the plane so that the vertex set V consists of points in general position and the edge set E consists of straight line segments between points of V. Two edges of a geometric graph are said to be *parallel*, if they are opposite sides of a convex quadrilateral.

In this paper we show that, for any fixed $k \geq 3$, any geometric graph on n vertices with no k pairwise parallel edges contains at most $O(n)$ edges, and any geometric graph on n vertices with no k pairwise crossing edges contains at most $O(n \log n)$ edges. We also prove a conjecture of Kupitz that any geometric graph on n vertices with no pair of parallel edges contains at most $2n - 2$ edges.

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1 Introduction

A *geometric graph* is a graph $G = (V, E)$ drawn in the plane so that the vertex set V consists of points in general position and the edge set E consists of straight line segments between points of V. See [9] for a survey of results about geometric graphs. Two edges of a geometric graph are said to be *parallel*, if they are opposite sides of a convex quadrilateral. Two edges *cross*, if their relative interiors intersect.

Pach and Törőcsik [11] proved that any geometric graph on n vertices with no k pairwise disjoint edges contains at most $(k-1)^4 n$ edges. In particular, their result implies that for any fixed $k \geq 2$ any geometric graph on n vertices with no k pairwise disjoint edges contains at most $O(n)$ edges. In this paper we show that the number of edges is at most linear in n also in a more general case.

Theorem 1 *Let $k \geq 2$ be a constant. Then any geometric graph on n vertices with no k pairwise parallel edges has at most $O(n)$ edges.*

The case $k = 2$ of Theorem 1 was considered first by Kupitz [7] who constructed, for any $n \geq 4$, a geometric graph with n vertices and $2n - 2$ edges containing no pair of parallel edges. Kupitz [7] also conjectured that the lower bound $2n - 2$ given by his construction is tight. A nearly tight upper bound $2n - 1$ was shown by Katchalski and Last [5, 8]. In this paper we show a refinement of the proof of Katchalski and Last [5, 8] giving Kupitz's conjecture.

Theorem 2 *For $n \geq 4$, any geometric graph on n vertices with no pair of parallel edges contains at most $2n - 2$ edges.*

A related question is how many edges can be contained in a geometric graph with no k pairwise crossing edges. For $k = 2$, Euler's formula gives the upper bound $3n - 6$ ($n \geq 3$). Pach, Sharokhi, and Szegedy [10] proved that any geometric graph on n vertices with no k pairwise crossing edges contains at most $O(n \log^{2k-4} n)$ edges ($k \geq 3$). This bound was improved to $O(n \log^{2k-6} n)$ in [2] ($k \geq 3$) (thus, in particular, to $O(n)$ for $k = 3$). For $k \geq 4$ we further improve this bound to $O(n \log n)$.

Theorem 3 *Let $k \geq 3$ be a constant. Then any geometric graph on n vertices with no k pairwise crossing edges has at most $O(n \log n)$ edges.*

One can ask about the dependence of the estimates in Theorems 1 and 3 on k. Our proof shows that the bound $O(n)$ in Theorem 1 could be replaced by the upper booound $c_k n$, where $1/c_k$ is exponential in k^3. Similarly, the bound $O(n \log n)$ in Theorem 3 could be replaced by the upper booound $c_k n \log n$, where $1/c_k$ is exponential in k^3. These bounds follow from the upper bound $d_k n$ ($1/d_k$ exponential in k) in Theorem 4, which can be obtained from the proof of Theorem 4 in [6]. However, it is possible that the bound in Theorem 4 can be improved to $e_k n$, where $1/e_k$ is polynomial in k. It would give the above bounds in Theorems 1 and 3 with $1/c_k$ polynomial in k.

Theorem 3 is derived from Theorem 1. In the proof of Theorem 1 we apply a projection method of Katchalski and Last [5, 8], Dilworth's theorem, and results on generalized Davenport–Schinzel sequences. Theorem 1 is proved in Section 2, Theorem 2 in Section 3, and Theorem 3 in Section 4.

2 Geometric graphs with no k pairwise parallel lines

2.1 Generalized Davenport–Schinzel sequences

For $l \geq 1$, a sequence is called *l-regular*, if any at most l consecutive terms are pairwise different. For $l \geq 2$, a sequence

$$S = s_1, s_2, \ldots, s_{3l-2}$$

of length $3l-2$ is said to be *of type* up-down-up(l), if the first l terms are pairwise different and, for $i = 1, 2, \ldots, l$,

$$s_i = s_{2l-i} = s_{(2l-2)+i}.$$

Thus, a sequence is of type up-down-up(2) if and only if it is an alternating sequence of length 4. It is well–known [3] (and not difficult to prove) that any 2-regular sequence over an n-element alphabet containing no alternating subsequence of length 4 has length at most $2n-1$. In the proof of Theorem 1 we apply the following related result.

Theorem 4 (Klazar and Valtr [6]) *Let $l \geq 2$ be a constant. Then the length of any l-regular sequence over an n-element alphabet containing no subsequence of type* up-down-up*(l) has length at most $O(n)$.*

2.2 Proof of Theorem 1

Let $G = (V, E)$ be a geometric graph on n vertices, with no k pairwise parallel edges. Let $V = \{v_1, v_2, \ldots, v_n\}$. W.l.o.g., we assume that no two points lie on a horizontal line.

Let $e \in E$. An oriented edge $\vec{e}$ is defined as the edge e oriented upwards. The *direction of* e, $\mathrm{dir}(e)$, is defined as the direction of the vector $\overrightarrow{v_i v_j}$, where $\vec{e} = (v_i, v_j)$. Let $E = \{e_1, e_2, \ldots, e_m\}$, where

$$0 < \mathrm{dir}(e_1) < \mathrm{dir}(e_2) < \cdots < \mathrm{dir}(e_m) < \pi$$

(if necessary, we perturb the vertices of G to make the directions of edges of G pairwise different).

Let P_1 and P_2 be the sequences of m integers obtained from the sequence

$$\vec{e_1}, \vec{e_2}, \ldots, \vec{e_m}$$

by replacing each edge $\vec{e_k} = (v_i, v_j)$ by integer i and by integer j, respectively. We call the sequences P_1, P_2 the *pattern sequences of* G. Inspired by [5, 8], our proof of Theorem 1 is based on a careful analysis of the pattern sequences.

Lemma 5 *For each $l \geq 1$, at least one of the pattern sequences P_1, P_2 contains an l-regular subsequence of length at least $|E|/(4l) = m/(4l)$.*

Lemma 6 *Neither of the pattern sequences P_1, P_2 contains a subsequence of type* up-down-up*(k^3).*

Before proving Lemmas 5 and 6 we complete the proof of Theorem 1.

Proof of Theorem 1. According to Lemma 5, at least one of the sequences P_1, P_2 contains a k^3-regular subsequence S of length at least $|E|/(4k^3)$. According to Lemma 6, the sequence S contains no subsequence of type up-down-up(k^3). Theorem 4 implies that the length of S is at most $O(n)$. Consequently,

$$|E| \leq 4k^3 \cdot O(n) = O(n). \qquad \Box$$

It remains to prove Lemmas 5 and 6.

2.3 Proof of Lemma 5

We apply a simple greedy algorithm [1] which, for given integer $l \geq 1$ and finite sequence A, returns an l-regular subsequence $B(A, l)$ of A. In the first step, an auxilliary sequence B is taken empty. Then the terms of A are considered one by one from left to right, and in each step the cosidered term is placed at the end of B iff this doesn't violate the l-regularity of B. Finally, the obtained l-regular subsequence B of A is taken for $B(A, l)$.

E.g., if $A = 1, 3, 1, 3, 5, 2, 2, 5, 1, 5, 1, 2$ and $l = 3$, then the algorithm returns the sequence $B(A, 3) = 1, 3, 5, 2, 1, 5, 2$. We prove Lemma 5 by showing that, given $l \geq 1$, at least one of the sequences $B(P_1, l)$, $B(P_2, l)$ obtained by the algorithm has length at least $|E|/(4l) = m/(4l)$.

Let $l \geq 1$ be given. For $i = 1, 2$ and for $1 \leq j_1 \leq j_2 \leq m$, let $P_{i,[j_1,j_2]}$ be the subsequence of P_i obtained from P_i by removing all terms preceding the j_1th term and following the j_2th term. Thus, $P_{i,[j_1,j_2]}$ is a continuous part of P_i of length $j_2 - j_1 + 1$.

Let $|T|$ denote the length of a sequence T, and let $I(T)$ denote the set of integers appearing in T.

Claim 7 *For each* $j = 1, 2, \ldots, m$*,*

$$\sum_{i=1}^{2} |B(P_{i,[1,j]}, l)| \geq j/(2l).$$

Proof. First, consider two integers j_1, j_2 such that $1 \leq j_1 \leq j_2 \leq m$. Obviously,

$$\{\overrightarrow{e_{j_1}}, \overrightarrow{e_{j_1+1}}, \ldots, \overrightarrow{e_{j_2}}\} \subseteq \{(v_a, v_b) | a \in I(P_{1,[j_1,j_2]}), b \in I(P_{2,[j_1,j_2]})\}.$$

and

$$|\{\overrightarrow{e_{j_1}}, \overrightarrow{e_{j_1+1}}, \ldots, \overrightarrow{e_{j_2}}\}| \le |\{(v_a, v_b) | a \in I(P_{1,[j_1,j_2]}), b \in I(P_{2,[j_1,j_2]})\}|.$$

Consequently,

$$j_2 - j_1 + 1 \le |I(P_{1,[j_1,j_2]})| \cdot |I(P_{2,[j_1,j_2]})|.$$

By the inequality between algebraic and geometric means,

$$\frac{\sum_{i=1}^{2} |I(P_{i,[j_1,j_2]})|}{2} \ge \sqrt{j_2 - j_1 + 1} \tag{1}$$

We now can prove the claim by induction on j. If $j \le \min\{16l^2, m\}$, then by (1) and by $j \le 16l^2$

$$\begin{aligned} \sum_{i=1}^{2} |B(P_{i,[1,j]}, l)| &\ge \sum_{i=1}^{2} |I(P_{i,[1,j]})| \\ &\ge 2\sqrt{j} \\ &\ge j/(2l). \end{aligned}$$

Suppose now that $16l^2 < j_0 \le m$ and that Claim 7 holds for $j = 1, 2, \ldots, j_0 - 1$. Since for $i = 1, 2$ each integer of $I(P_{i,[j_0-4l^2+1, j_0]})$ not appearing among the last $l-1$ terms in $B(P_{i,[1,j_0-4l^2]}, l)$ appears more times in $B(P_{i,[1,j_0]}, l)$ than in $B(P_{i,[1,j_0-4l^2]}, l)$, we have

$$|B(P_{i,[1,j_0]}, l)| \ge |B(P_{i,[1,j_0-4l^2]}, l)| + |I(P_{i,[j_0-4l^2+1,j_0]})| - (l-1).$$

Consequently, by the inductive hypothesis and by (1),

$$\begin{aligned} \sum_{i=1}^{2} |B(P_{i,[1,j_0]}, l)| &\ge (j_0 - 4l^2)/(2l) + 2\sqrt{4l^2} - 2(l-1) \\ &> j_0/(2l). \qquad \square \end{aligned}$$

Proof of Lemma 5. Lemma 5 easily follows from Claim 7 (with $j = m$) and from the pigeon-hole principle. $\square$

2.4 Proof of Lemma 6

In the proof of Lemma 6 we apply the folowing easy consequence of Dilworth's theorem [4].

Theorem 8 *If the union of three partial orderings on a set I of size at least $(k-1)^3+1$ is a linear ordering on I, then at least one of the partial orderings contains a chain of length k.*

Proof. Let $\preceq_1$, $\preceq_2$, $\preceq_3$ be the three partial orderings on I. If $(I, \preceq_1)$ doesn't contain a chain of length k then, by Dilworth's theorem, it can be covered by at most $k-1$ antichains. Consequently, there is an antichain A of size $(k-1)^2+1$ in $(I, \preceq_1)$. If we restrict our attention to A and to orderings $\preceq_2$, $\preceq_3$, another application of Dilworth's theorem gives k elements in A which form a chain with respect to $\preceq_2$ or to $\preceq_3$. □

Proof of Lemma 6. Because of symmetry, it suffices to prove Lemma 6 for the pattern sequence P_1. Suppose to the contrary that P_1 contains a subsequence of type up-down-up(k^3). Thus, there is a subsequence

$$S = s_1, s_2, \ldots, s_{3k^3-2}$$

of P_1 such that the integers $s_1, s_2, \ldots, s_{k^3}$ are pairwise different and that, for $i = 1, 2, \ldots, k^3$, $s_i = s_{2k^3-i} = s_{(2k^3-2)+i}$. For simplicity of notation, suppose that $s_i = i$ $(i = 1, \ldots, k^3)$ and that $S = P_{1,[1,3k^2-2]}$. We obtain a contradiction by showing that k of the edges $e_1, e_2, \ldots, e_{3k^3-2}$ are pairwise parallel.

Define three partial orderings $\preceq_1$, $\preceq_2$, $\preceq_3$ on the set $I = \{1, 2, \ldots, k^3\}$ as follows.

Definition 9 *Let $i, j \in I$, and let $\mathrm{dir}(\overrightarrow{v_i v_j})$ denote the direction of the vector $\overrightarrow{v_i v_j}$. Then*

(i) $i \prec_1 j$, if $i < j$ and $\mathrm{dir}(\overrightarrow{v_i v_j}) \in [\mathrm{dir}(\overrightarrow{e_{k^3}}), \pi)$,

(ii) $i \prec_2 j$, if $i < j$ and $\mathrm{dir}(\overrightarrow{v_i v_j}) \in (\pi, \pi + \mathrm{dir}(\overrightarrow{e_{2k^3-1}}))$,

(iii) $i \prec_3 j$, if $i < j$ and $\mathrm{dir}(\overrightarrow{v_i v_j}) \in [\pi + \mathrm{dir}(\overrightarrow{e_{2k^3-1}}), 2\pi) \cup (0, \mathrm{dir}(\overrightarrow{e_{k^3}}))$.

Since the union of $\preceq_1$, $\preceq_2$, $\preceq_3$ is a linear ordering on I, Theorem 8 implies that one of the orderings $\preceq_1$, $\preceq_2$, $\preceq_3$ contains a chain $i_1, i_2, \ldots, i_k$ of length k. We distinguish the corresponding three possible cases.

If $i_1 \prec_1 i_2 \prec_1 \cdots \prec_1 i_k$, then the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ are pairwise parallel. Indeed, if $1 \le j < j' \le k$, then the inequalities

$$0 \le \mathrm{dir}(\overrightarrow{e_{i_j}}) < \mathrm{dir}(\overrightarrow{e_{i_{j'}}}) \le \mathrm{dir}(\overrightarrow{e_{k^3}}) \le \mathrm{dir}(\overrightarrow{v_{i_j} v_{i_{j'}}}) < \pi$$

show that the edges $e_{i_j}, e_{i_{j'}}$ are parallel (see Fig. 1).

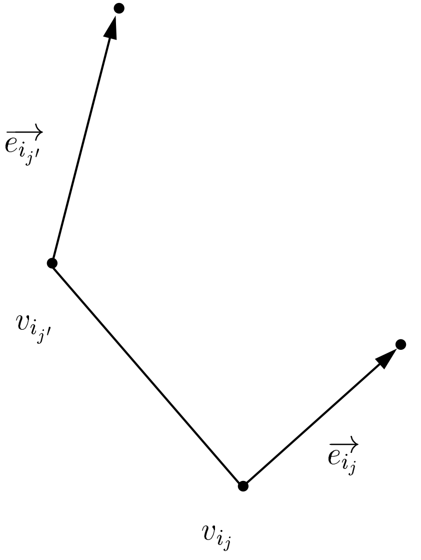

Figure 1: The edges $e_{i_j}, e_{i_{j'}}$ are parallel.

Similarly, if $i_1 \prec_2 i_2 \prec_2 \cdots \prec_2 i_k$ then the edges $e_{(2k^3-2)+i_1}$, $e_{(2k^3-2)+i_2}, \ldots, e_{(2k^3-2)+i_k}$ are pairwise parallel, and if $i_1 \prec_3 i_2 \prec_3 \cdots \prec_3 i_k$ then the edges $e_{2k^3-i_1}, e_{2k^3-i_2}, \ldots, e_{2k^3-i_k}$ are pairwise parallel. $\square$

3 Proof of Kupitz's conjecture

In this section we prove Theorem 2. Our proof is based on the proof [5] of a weaker form of Theorem 2 (with $2n - 2$ replaced by $2n - 1$).

In the proof we apply the following modification of Lemma 2 in [5].

Lemma 10 *Let A_i, A_j, A_k, A_ℓ be four points appearing in this order on a convex curve γ. Let P, Q be two points inside γ. Consider the four (closed) segments*

$$PA_i, QA_j, PA_k, QA_\ell$$

and assume that among them there is no segment s such that s contains only one of the points P, Q and the line supporting s contains both of them. Then two of the four segments are in convex position.

Lemma 10 can be proved analogously as Lemma 2 in [5] (in fact, Lemma 2 in [5] could be replaced by our Lemma 10).

Proof of Theorem 2. Let G be a geometric graph on n vertices with no pair of edges in convex position. We use the definitions and notation from [5]. The following bound on the number of edges in G was shown in [5]:

$$e \leq 2n - 1.$$

We need to show that

$$e < 2n - 1.$$

Suppose to the contrary that $e = 2n - 1$. It then follows from the proof in [5] that each vertex v_i of G has a leftmost edge $v_i v_{l(i)}$ and a rightmost edge $v_i v_{r(i)}$ ($l(i) \neq r(i)$), and that the length of the pattern sequence $PS(G)$ is exactly $2n - 2$ (see [5] for definitions).

For each vertex v_i of G, we define an interval $I(v_i)$ on C by

$$I(v_i) = C \cap \operatorname{conv}\left(\overrightarrow{v_i v_{l(i)}} \cup \overrightarrow{v_i v_{r(i)}}\right).$$

Certainly, $I(v_i)$ contains all points of $D(G)$ colored by color i.

Observation 11 *The intervals $I(v_i), i = 1, \ldots, n$, form a nested set, i.e., if two of them intersect then one of them is contained in the other one.*

Proof. If two intervals $I(v_i), I(v_j), i \neq j$, intersect and none of them is contained in the other one, then the points $\alpha_{il(i)}$, $\alpha_{jl(j)}$, $\alpha_{ir(i)}$, $\alpha_{jr(j)}$ appear in this order on C, and Lemma 10 implies that two of the edges $v_iv_{l(i)}$, $v_iv_{r(i)}$, $v_jv_{l(j)}$, $v_jv_{r(j)}$ are in convex position, a contradiction. $\square$

Let v_k be a vertex of G such that the angle between $\overrightarrow{v_kv_{r(k)}}$ and $\overrightarrow{v_kv_{l(k)}}$ is maximal. Consider the two points $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$. They both lie outside $I(v_k)$, since the angle between $\overrightarrow{v_kv_{r(k)}}$ and $\overrightarrow{v_kv_{l(k)}}$ is smaller than π.

Observation 12 *No interval $I(v_i)$ contains both $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$.*

Proof. If an interval $I(v_i)$ contains both $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$, then, by the maximality of the angle between $\overrightarrow{v_kv_{r(k)}}$ and $\overrightarrow{v_kv_{l(k)}}$, one of the rays $\overrightarrow{v_iv_{r(i)}}$, $\overrightarrow{v_iv_{l(i)}}$ intersects one of the rays $\overrightarrow{v_{r(k)}v_k}$, $\overrightarrow{v_{l(k)}v_k}$ outside C, and the corresponding two edges of G are in convex position. $\square$

It follows from Observation 12 that there are at least three maximal intervals $I(v_i)$ (the interval $I(v_k)$ and the two maximal intervals containing $\alpha_{r(k)k}$ and $\alpha_{l(k)k}$, respectively).

By Observation 11, maximal intervals $I(v_i)$ are pairwise disjoint.

Observation 13 *Each maximal interval $I(v_i)$ containing m_i intervals $I(v_{i'})$ (incl. the interval $I(v_i)$ itself) contains at most $2m_i - 1$ points of $D(G)$.*

Proof. The part of $PS(G)$ corresponding to $I(v_i) \cap D(G)$ is a Davenport–Schinzel sequence of order 2 on m_i integers (i.e., it contains no alternating subsequence of length 4). As we have already mentioned in Section 2.1, it is well-known [3] that the length of such a sequence is at most $2m_i - 1$. $\square$

Thus, the length of $D(G)$ (and also of $PS(G)$) is at most

$$2n - (\sharp \text{ of maximal intervals } I(v_i)) \leq 2n - 3,$$

a contradiction. This completes the proof of Theorem 2. $\square$

4 Graphs with no k pairwise crossing edges

Here we show Theorem 3. Let $k \geq 2$ be a constant, and let $f_k(n)$ be the maximum number of edges in a geometric graph on n vertices with no k pairwise crossing edges. Let $G = (V, E)$ be a geometric graph on n vertices with no k pairwise crossing edges. Introduce Cartesian coordinate system so that the y-axis partitions V into two parts which are as equal as possible, thus the sets

$V^- = \{v \in V | \text{the } x\text{-coordinate of } v \text{ is negative}\}$,

$V^+ = \{v \in V | \text{the } x\text{-coordinate of } v \text{ is positive}\}$

have sizes

$$|V^-| = \lfloor n/2 \rfloor, \qquad |V^+| = \lceil n/2 \rceil .$$

Partition E into three subsets E^+, E^-, E' such that E^+ contains the edges with both endpoints in V^+, E^- contains the edges with both endpoints in V^-, and E' contains the edges crossing the y-axis.

To obtain a bound on the size of E', consider the mapping T given by $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x})$ $(x \neq 0)$. Further, consider the graph $\tilde{G}$ on the vertex set $\tilde{V} = \{T(v) | v \in V\}$ with two vertices $T(v), T(w)$ connected by an edge if and only if $\{v, w\} \in E'$. The graph $\tilde{G}$ contains no k pairwise parallel edges, since otherwise the corresponding k edges in E' would be pairwise crossing. By Theorem 1, $\tilde{G}$ contains $O(n)$ edges. Consequently,

$$|E'| = O(n).$$

Obviously,

$$|E^-| \leq f_k(\lfloor n/2 \rfloor), \qquad |E^+| \leq f_k(\lceil n/2 \rceil).$$

Thus,

$$f_k(n) \leq f_k(\lfloor n/2 \rfloor) + f_k(\lceil n/2 \rceil) + O(n).$$

Consequently,

$$f_k(n) = O(n \log n). \qquad \Box$$

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