THIRD ANNUAL DONET MEETING
1996

ed. M. Klazar

Contents

1 Preface

Third Annual Meeting of the Discrete Optimization Network (DONET) was held at Štiřín Castle in the Prague Area, May 19-24, 1996. This was the last of the DONET Annual meetings within the present framework of DONET. The previous annual meetings were held in Trento in 1993 and in London in 1995. We enclose shorter abstracts of most of the talks presented at the meeting. We are also happy to include extended abstracts and short papers covering most of the invited talks. Perhaps this, together with some photos and (a few memories) will provide us with an encouragement to the future. One can see that DONET contributed in an essential way to the development of Discrete Optimization in Europe, especially by providing opportunities for the critical group of young researchers. We thank to the founder of the project prof. B. Korte (Bonn) and the present acting director of DONET project prof. J. Fonlupt (Paris) for all their energy in starting and managing this project. And we hope that we have enough luck to be able to continue and deepen our activity in similar framework in future.

Jaroslav Nešetřil

Carsten Thomassen, Technical University Danemark (non-Donet participant)
Zsolt Tuza, Eotvos University, Budapest
Pavel Valtr, Charles University, Prague
Jens Vygen, University of Bonn
Dominic Welsh, University of Oxford

Winfried Hochstättler, University of Koeln
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Stan van Hoesel, University of Limburg
Hein van der Holst, CWI Amsterdam
Tibor Jordan, Eötvös University, Budapest
Anjai Kapoor, University of Padova
Judith Keijsper, University of Amsterdam
Kyriakos Kilakos, LSU, London
Martin Klazar, Charles University, Prague
Bernhard Korte, University of Bonn
Jan Kratochvíl, Charles University, Prague
Monique Laurent, LIENS, Paris
Robert Leese, University of Oxford
Martin Loebl, Charles University, Prague
András Lukács, Eötvös University, Budapest
Colin McDiarmid, University of Oxford
Criel Merino, University of Oxford
Raffaele Mosca, University La Sapienza, Rome
Denis Naddef, University Joseph Fourier, Grenoble
Jaroslav Nešetřil, Charles University, Prague
Steven Noble, University of Oxford
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Andre Rohe, University of Bonn
Irasema Sarmiento, University of Oxford
András Sebő, University Joseph Fourier, Grenoble
Bruce Shepherd, LSU, London
Mark Shepherd, University of Oxford
Bruno Simeone, University La Sapienza, Rome
Stefano Smriglio, University La Sapienza, Rome
Leen Stougie, University of Amsterdam
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2 List of participating institutions and the local coordinators

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Padua	M. Comforti	*University of Padua*
Paris	J. Fonlupt	*Université Pierre et Marie Curie* *Tour 46–56 — 5ème étage* *4 Place Jussien* *75005 Paris* *France*
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16 List of participants

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Christopher Anhalt, University of Bonn
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Jorgen Bang-Jensen, Odense University, Denmark
(non-Donet participant)
Eric Bartels, University of Oxford
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Adrian Bondy, University of Lion
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Ondřej Čepek, Charles University, Prague
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Matthias Hayer, University of Koeln
Jan van den Heuvel, LSU, London
Petr Hliněný, Charles University, Prague

exact computation of each of which is well known to be $\#P$-hard. Here I survey the problem of finding a fully polynomial randomised approximation schemes for approximating the value of $T(G;x,y)$. In particular I shall describe recent results obtained with Noga Alon and Alan Frieze which will give an fpras for evaluating T for any dense graph G, that is, any graph on n vertices whose minimum degree is $\Omega(n)$, whenever $x \geq 1$ and $y \geq 1$, and in various additional points. This region includes evaluations of reliability and partition functions of the ferromagnetic Q-state Potts model, and extends to linear matroids where T specialises to the weight enumerator of linear codes.

3 André Bouchet
Multimatroids. Tightness and fundamental graphs

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Consider a partition Ω of a finite set U. A *subtransversal* (resp. *transversal*) of Ω is a subset $A \subseteq U$ such that $|A \cap \omega| \leq 1$ (resp. $|A \cap \omega| = 1$) holds for all ω in Ω. We denote by $\mathcal{S}(\Omega)$ the set of subtransversals of Ω.
A *multimatroid* is a triple $Q = (U, \Omega, r)$ with a partition Ω of a finite set U and a *rank function* $r : \mathcal{S}(\Omega) \to \mathcal{N}$ satisfying the four following axioms:

1 $r(\emptyset) = 0$*;*

2 $r(A) \leq r(A+x) \leq r(A)+1$ *is satisfied if* A *is a subtransversal of* Ω*,* x *is an element of* U*, and* $A+x$ *is a subtransversal;*

3 Submodularity inequality: $r(A)+r(B) \geq r(A \cup B)+r(A \cap B)$ *is satisfied if* A*,* B *and* $A \cup B$ *are subtransversals;*

4 $r(A+x) - r(A) + r(A+y) - r(A) \geq 1$ *is satisfied if* A *is a subtransversal of* Ω*,* $\{x, y\}$ *is a pair contained in a class* ω *of* Ω*, and* A *is disjoint from* ω*.*

An *independent set* is a subtransversal I such that $r(I) = |I|$, a *base* is a maximal independent set, and a *circuit* is a subtransversal that is not independent and minimal with this property. If each skew class has cardinality equal to the positive integer q then Q is also called a *q-matroid*.
The multimatroid Q is *nondegenerate* if every class has at least two elements. Then it is easy to prove, by using the fourth axiom, that

every base is a transversal. A basic result of the theory of multimatroids says that the union of every base with every class of Ω contains at most one circuit. The multimatroid Q is said to be *tight* if it is non-degenerate and the union of a base with a class of Ω always contains a circuit, called *fundamental circuit*.
The fundamental circuits, with respect to matroids and tight multimatroids, play similar roles. In particular they can be used to define a fundamental graph $G_Q(B)$ associated to every base B of Q. This graph carries a great part of the information concerning the connectivity of Q. The first basic result says that Q is connected if and only if $G_Q(B)$ is connected. Here a multimatroid is said to be *connected* if, for every bipartition $\{X_1, X_2\}$ of U such that X_1 and X_2 are unions of classes of Ω, there exists a circuit C such that $C \cap X_1 \neq \emptyset \neq C \cap X_2$.
At a deeper level one can extend, by specifying the transformations of fundamental graphs when the base is modified, some relations between minors and connectivity. Here the *minor* of a multimatroid Q, with respect to an element x contained in the class ω of Ω, is the multimatroid Q' defined by the rank function $r'(A) = r(A+x) - r(x)$, for $A \in \mathcal{S}(\Omega - \omega)$. Thus, if Q is a tight q-matroid, we define q minors of Q with respect to the q elements in ω. We prove that at least $q-1$ of these minors are connected. This extends a theorem of TUTTE in matroid theory, saying that, if we contract and delete one element in a connected matroid, then at least one of the two minors is connected.
There is a notion of linear representability for a multimatroid. We have given, with DUCHAMP, a characterization of the binary 2-matroids by means of two excluded minors. GASSE has recently proved that, if a connected tight 2-matroid Q is not binary, then every element belongs to a minor equal to one of the two forbidden minors. This extends a theorem of SEYMOUR in matroid theory. An extension of a theorem of LEHMANN, saying that a connected matroid is determined by the set of its circuits containing a given element, has also been extended to tight multimatroids.
To complete this abstract we note that the preceding properties become false with multimatroids that are not tight.

linearly in n. Our approach is based on a new, and computationally efficient approach for reconstructing phylogenetic trees from aligned DNA sequences.

Jens Vygen

A Linear–Time Algorithm for Partitioning 2-Dimensional Pointsets

University of Bonn

At the DONET Conference 1995 in London I introduced a theorem describing the structure of the optimum solutions of a crucial problem during VLSI-placement. I mentioned that this leads to a linear–time algorithm at least for a special case. Today I can show how the results can be extended to the general case.
The mathematical description of the problem is as follows: Given a finite set S of points in the plane, weights $w : S \to R_+$, and capacities $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ of the four quadrants, we are looking for a fractional partition $g : S \times \{0,1,2,3\} \to [0,1]$ with $\sum_{i=0}^{3} g(s,i) = 1$ for all $s \in S$, such that the capacity constraints $\sum_{f(s)=i} w(s) \leq \kappa_i$ $(i = 0,1,2,3)$ are met and the total movement $\sum_{s \in S} \sum_{i=0}^{3} w(s) g(s,i) d(s, Q_i)$ is minimized. Here Q_i denotes the i-th quadrant and d denotes the L_1-distance.

Dominic Welsh

Polynomial time randomised approximation schemes for Tutte-Gröthendieck invariants

University of Oxford

The Tutte polynomial $T(G; x, y)$ of a graph G encodes numerous interesting combinatorial quantities associated with the graph. Its evaluation in various points in the (x, y) plane give the number of spanning forests of the graph, the number of its strongly connected orientations, the number of its proper k-colourings, the (all terminal) reliability probability of the graph, and various other invariants the

Bruno Simeone

Generation of all k-partitions of an n-set in amortized constant time per partition

University La Sapienza, Rome

This is a joint work with Fabio Grasso. The present paper deals with the sequential generation of all k-partitions of an n-set, for a fixed $1 \leq k \leq n$. The fastest known algorithm for this task is due to Ehrlich: it takes $O(k)$ time per partition. Here we propose another algorithm which takes only amortised constant time per partition. The algorithm is a modification of a previous procedure described by Nijenhuis and Wilf. We give an implementation which makes use of simple data structures, requires only $2n + k + c$ memory locations, where c is a small constant, and achieves the above time bound. Such bound is established through a careful analysis of the changes between any two consecutive partitions.

L. A. Székely

The number of nucleotide sites needed to accurately reconstruct large evolutionary trees

Eotvos University, Budapest

This is joint work with M. A. Steel and P. L. Erdős. Biologists seek to reconstruct evolutionary trees for increasing numbers of species, n, from aligned genetic sequences. How fast the sequence length N must grow, as a function of n, in order to accurately recover the underlying tree with probability $1 - \epsilon$, if the sequences evolve according to simple stochastic models of nucleotide substitution? We show that for certain models, the sequence length N can grow surprisingly slowly with n (sublinearly for a wide range parameters, and even as a power of $\log n$ in a narrow range, which roughly meets the lower bound from information theory). By contrast, a more traditional technique (maximum compatibility) provably requires N to grow faster than

4 Peter J. Cameron Fixed Points and Derangements

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Abstract

This note concerns numbers of fixed points of elements of a permutation group. A simple transformation relates the probabiliity generating function for the number of fixed points of a random group element with the exponential generating function for the number of orbits of the group on ordered tuples. For infinite groups, this no longer applies. The formalism often produces a number which in the finite case would be the proportion of derangements; in the infinite case its meaning is usually not clear.

The article concludes with a result of Parker extending the orbit-counting lemma from fixed points to cycles. One open question concerns whether Block's Lemma extends to this situation.

4.1 Fixed points

Throughout the following, G is a permutation group on a set Ω, which may be finite or infinite. (The cardinality of Ω is the *degree* of G.) The basic tool relating fixed points to orbits is the *orbit-counting lemma* (mis-attributed to Burnside):

Theorem 4.1.1 *Let G have finite degree; let $c(g)$ be the number of fixed points of $g \in G$. Then the number of orbits of G is given by*

$$\frac{1}{|G|} \sum_{g \in G} c(g).$$

The well-known proof involves counting edges in the bipartite graph on $\Omega \times G$, where $x \in \Omega$ is joined to $g \in G$ if and only if g fixes x. Mark Jerrum has pointed out that a random walk on this bipartite graph can be used to choose a random G-orbit from the uniform distribution.
The following consequence of the orbit-counting lemma was pointed out to me by Nigel Boston. Let G be a permutation group of finite degree n. Let $X(t)$ be the probability generating function for the number of fixed points of a random element of G: thus,

$$X(t) = \frac{1}{|G|} \sum_{i=0}^{n} f_i t^i,$$

where f_i is the number of elements of G which have exactly i fixed points. Now let m_i be the number of orbits of G in its induced action on i-tuples of distinct elements of Ω, and $Y(t)$ the exponential generating function for this sequence; that is,

$$Y(t) = \sum_{i=0}^{n} \frac{m_i t^i}{i!}.$$

Theorem 4.1.2

$$Y(t) = X(t+1).$$

Proof. If an element g fixes i points, then it fixes $(i)_j = i(i-1)\cdots(i-j+1)$ ordered j-tuples. So

$$m_j = \frac{1}{|G|} \sum_{i=j}^{n} f_i (i)_j.$$

The result follows by simple manipulation.

A *derangement* is a permutation with no fixed points. Thus, the proportion of derangements in G is $f_i = |G| X(0)$, and we have:

Corollary 4.1.3 *With the above notation, the probability that a random element of G is a derangement is $Y(-1)$.*

Question. What (linear or non-linear) constraints are satisfied by the vector $(f_0/|G|, \ldots, f_m/|G|)$?

valid inequalities for that polytope say. I will end by my favorite conjecture on that polytope.

Rudi Pendavingh

A Polynomial Time Algorithm For Bibranching

University of Amsterdam

This is joint work with J. Keijsper. Let $D = (V, A)$ be a digraph, $W \subset V$ a subset of its vertices. A *bibranching* is a set $B \subset A$ such that

1. for all $v \in W$, B contains a directed path from v to a vertex in $V \setminus W$, and
2. for all $v \in V \setminus W$, B contains a directed path from a vertex in W to v.

For $W = \{r\}$, a bibranching is exactly an r-branching, i.e. a directed tree rooted at r. Given a bipartite graph, let W be one of the colour classes and orient all arcs away from W. A set of arcs is a bibranching of the resulting digraph iff it is an edge cover of the original graph. Now let w be a positive integer weight function on the arrow set. Then, the linear programming duality equation for minimum weight bibranching has integral optimum solutions. We will present a primal-dual algorithm for constructing a bibranching of minimum weight in polynomial time.

Bruce Shepherd

A Node capacitated routing problem arising in local optical networks

LSU, London

We discuss several problems arising in network routing and design. In particular we consider a node-capacitated routing problem related to ring networks of passive optical networks.

Kyriakos Kilakos

Polytopes of Bounded Stable Sets

LSU London

The *t-stable set polytope* of a graph is the convex hull of incidence vectors of stable sets of cardinality at most t in the graph. For $t = 3$, we describe the facet defining inequalities of the t-stable set polytope of any bipartite graph G. In addition, we show that for any integral weighting on the vertices of G, the chromatic number of G (with respect to this weighting) is equal to the round up of its fractional chromatic number plus one.

More generally, for any integer t, we discuss facet defining inequalities of the t-stable set polytope of bipartite graphs.

Colin McDiarmid

Colouring Proximity Graphs in The Plane

University of Oxford

Given a set S of points in the plane, and given $d > 0$, let $G(S, d)$ denote the graph with vertex set S and with distinct vertices adjacent whenever the Euclidean distance between them is less than d. We are interested in colouring such 'proximity' or 'interference' graphs. One application where this problem arises is in the design of mobile telephone networks, when we need to assign radio frequency bands (colours) to transmitters (points in S) to avoid interference. We find for example that, if the set S has density σ, then the chromatic number χ satisfies $\chi(G(S,d))/d^2 \to \sigma\sqrt{3}/2$ as $d \to \infty$. This is joint work with Bruce Reed.

Denis Naddef

Discovering The TSP Polytope

University Joseph Fourier, Grenoble

In this talk, which is inspired greatly by discussions with Yves Pochet of CORE, I will try to give you a feeling of what most of the known

That there are some non-trivial constraints is shown by the following result of Arjeh Cohen:

Theorem 4.1.4 *If G is transitive with degree n, then at least $|G|/n$ elements of G are derangements, with equality if and only if G is sharply 2-transitive. Moreover, the subgroup generated by the derangements is transitive and contains all elements of G whose number of fixed points is not equal to 1.*

This is shown by transferring from Y to X the following constraint satisfied by the numbers m_i of orbits on i-tuples: $m_2 \geq m_1(m_1 - 1) + m_1^*$, where m_1^* is the number of orbits on points with cardinality greater than 1. (A permutation group G is *sharply 2-transitive* if any 2-tuple of distinct points is mapped to any other by a unique element of G. Such finite groups have been completely determined by Zassenhaus in the 1930s.)

Now let G be a permutation group of infinite degree. There is no obvious way in which to speak of proportions of elements with a given number of fixed points. However, numbers of orbits can be counted. We say that G is *oligomorphic* if it has only finitely many orbits on i-tuples of distinct elements of Ω, for all non-negative integers i. If m_i is this number of orbits, then we can define the generating function $Y(t)$ as above. Often it turns out that $Y(-1)$ can be calculated, as we will see; the meaning of this number is not clear.

4.2 Examples

Example 1. First we look at the classical example, the number of derangements in the symmetric group. There are many entirely different ways of computing this number. The argument here, though, may be new. The symmetric group S_n has a unique orbit on i-tuples for $i \leq n$, and none for $i > n$. So $Y(t)$ is the exponential series $\sum t^i/i!$, truncated at the term $t^n/n!$. The proportion of derangements is obtained by substituting $t = -1$, and we obtain the familiar formula, and the fact that it is approximately $1/\mathrm{e}$.
Moreover, for the infinite symmetric group, the number of orbits on i-tuples is 1 for all i. So $Y(t) = \mathrm{e}^t$. This series converges for all t, and $Y(-1) = \mathrm{e}^{-1}$, the limiting value of the proportion of derangements in finite symmetric groups.

Example 2. Let G be the group of order-preserving permutations of the rational numbers (or of the real numbers). Then G is oligomorphic: it has $i!$ orbits on i-tuples, corresponding to the $i!$ different permutations relating the order of the i-tuple to the natural order in $\mathbb{Q}$ or $\mathbb{R}$. So

$$Y(t) = \sum_{i=1}^{\infty} t^i = \frac{1}{1-t}.$$

This would suggest that $Y(-1) = \frac{1}{2}$. More formally,

$$Y(-1) = 1 - 1 + 1 - 1 + \ldots ;$$

the series is not convergent, but can be summed by any number of methods to $\frac{1}{2}$. For example, the radius of convergence of $X(t)$ is 1, but there is a unique singularity on the unit circle (at 1), and $X(t)$ can be analytically continued to all other points of $\mathbb{C}$: its value at -1 is $\frac{1}{2}$.
Is there any sense in which half of the elements of this group are derangements?

Example 3. Let G be the group $S_r^{\{2\}}$, defined to be S_r acting on 2-sets, with degree $n = \binom{r}{2}$. Then m_i is the number of graphs with r unlabelled vertices and i labelled edges. Moreover, it can be shown (using results about the cycle distribution in random permutations) that

$$\begin{aligned} f_0/|G| &= \sum_{i=0}^{n} (-1)^i m_i / i! \\ &\to 2\mathrm{e}^{-3/2} = 0.446260\ldots \text{ as } r \to \infty. \end{aligned}$$

Moreover, we can take the limiting case to be $S^{\{2\}}$, the infinite symmetric group acting on 2-sets. Then $m_i(S_r^{\{2\}}) \to m_i(S^{\{2\}})$ as $r \to \infty$. Indeed, the number of graphs with r vertices and i edges increases until $r = 2i$ and then remains constant.

Question. Is there some sense of convergence in which the series $Y(-1)$ for the group $S^{\{2\}}$ converges to $2\mathrm{e}^{-3/2}$?

given set of points in the Euclidean plane, such that the sum of the direction changes at each vertex along the tour is minimized. This result has important consequences for the problem of matroid parity, as it resolves the long-standing open question about the hardness of the weighted linear matroid parity problem in the affirmative.
The Angular-Metric TSP is closely related to the problem "Angle-Restricted Tour" (ART): For a given set $A \subseteq (-\pi; +\pi]$ of angles, we have to decide whether a set P of n points in the Euclidean plane allows a closed directed tour consisting of straight line segments, such that all angles between consecutive line segments are from the set A. We present a variety of algorithmic and combinatorial results on this problem. In particular, we show that any finite set of at least five points in general position allows a "pseudoconvex" tour (i. e. a tour where all angles are nonnegative), and we derive a fast algorithm for constructing such a tour. Moreover, we give a complete classification (from the computational complexity point of view) for the special cases where the tour has to be part of the orthogonal grid.
The paper presented is joint work with Gerhard Woeginger (Graz).

Anjai Kapoor

Triangle Free Graphs That Are Odd-Signable

University of Padova

A graph is odd-signable if there exists an assignment of "odd" and "even" labels to the edges so that every hole has an odd number of odd edges. Graphs with no even holes are a subclass of this class. Triangle-free (TF) odd-signable graphs form a class of basic graphs for cap-free odd-signable graphs, which in turn are a basic class in the study of graphs with no even holes.
Here we give a decomposition theorem and recognition algorithm for TF odd-signable graphs. We also show that all TF odd-signable graphs can be constructed by a sequence of "ear additions". We make some conjectures regarding the description of the stable set polytope for these graphs.
These results were jointly obtained with Cornuejols, Conforti and Vuskovic.

It is balanced if no square submatrix of odd order contains exactly two ones per row and per column. In this talk, we survey results and open problems about these and related classes of matrices.

Bert Gerards

The Matroids Representable Over The 4-element Field

CWI, Amsterdam

Together with Jim Geelen and Ajai Kapoor, we present the complete list of forbidden minors for GF(4)-representable matroids.

Winfried Hochstättler

On Bases of Circuit Lattices

University of Koeln

This is joint work with Martin Loebl. Given a binary matrix B the circuit lattice of the corresponding binary matroid is the integer hull of the $0,1$-vectors in the kernel of B. We discuss a couple of observations concerning the following conjecture: The circuit lattice of a binary matroid has a basis of $0,1$-vectors or, in other words, the integer lattice of a binary code has a basis of codewords.
We relate this conjecture to a certain Sylvester-type Hadamard matrix M and prove a duality statement about submatrices of full row length.
Finally, we disprove a stronger version of the above conjecture, indicating, that it, if true, does not have a local character.

Sandor Fekete

Angle-Restricted Tours in the Plane

University of Koeln

Earlier this year, Aggarwal, Coppersmith, Khanna, Motwani, and Schieber showed that the so-called "Angular-Metric TSP" is NP-complete. In this problem, one has to find a Hamiltonian path for a

4.3 Parker's Lemma

The following result was proved by Richard Parker. It suggests a technique for obtaining information about an unknown permutation group by looking at the cycle structure of a random element. (This situation arises in computatioonal Galois theory: given an integer polynomial, we can compute the cycle structure of the Frobenius automorphism on the roots of its reduction modulo a prime.)

Theorem 4.3.1 (Parker's Lemma) *Let G be a finite permutation group of degree n. Let $c_i(g)$ be the number of i-cycles of the element $g \in G$. Let $\mathcal{C}_i$ be the set of all i-cycles which occur in the decomposition of some element of G (with G acting by conjugation on $\mathcal{C}_i$). Then the number of orbits of G on $\mathcal{C}_i$ is*

$$\frac{1}{|G|}\sum_{g \in G} ic_i(g).$$

The proof is ingenious but elementary. In particular, the Orbit-Counting Lemma (which is the special case $i = 1$, since every point occurs as a 1-cycle in the identity) is not used in the proof.
Parker's Lemma has the following curious consequence.

Corollary 4.3.2 *With the notation of Parker's Lemma, let $\mathcal{C} = \bigcup_i \mathcal{C}_i$ be the set of all cycles of elemens of G. Then the number of orbits of G on $\mathcal{C}$ is n.*

For $\sum_i ic_i(g) = n$ for all $g \in G$.

The *Parker vector* of G is the n-tuple $(a_1, \ldots, a_n)$, where a_i is the number of orbits of G on $\mathcal{C}_i$. At the two extremes, the trivial group has Parker vector $(n, 0, \ldots, 0)$, and the symmetric group has Parker vector $(1, 1, \ldots, 1)$. The components of the Parker vector need not be monotonic. If G is the cyclic group of order n acting regularly, its Parker vector has components

$$a_i = \begin{cases} \phi(i), & \text{if } i \text{ divides } n, \\ 0, & \text{otherwise,} \end{cases}$$

where ϕ is Euler's function.

Question. What is the convex hull of the set of Parker vectors of permutation groups of degree n?

Suppose that G is a group of automorphisms of an incidence structure of points and blocks, whose incidence matrix has 'full rank' (rank equal to the number of points). Examples of such structures include non-trivial 2-designs (balanced incomplete block designs). Then *Block's Lemma* asserts that G has at least as many orbits on blocks as on points. That is, the first component of the Paarker vector on blocks dominates that on points. The following generalisation seems unlikely, but I know of no counterexamples.

Question. Under the above hypothesis, is it true that the first n components of the Parker vector of G on blocks dominate those for G on points, where n is the number of points?

proved that the minimum number of components in a b-detachment of G equals maximum $\{b(X) + c(G - X) - |L(X)| : X$ is a subset of $V\}$, where $b(X)$ is the sum of $b(v)$ for $v \in X$, $c(G-X)$ is the number of components of $G - X$, and $L(X)$ is the set of edges of G with at least one end in X. We give a direct polytime algorithm for finding a b-detachment with a minimum number of components.

Michele Conforti

A Theorem of Truemper about Wheels and Three-Path Configurations

University of Padova

Suppose one has to "sign" the edges of a graph with even an odd labels so that, for each chordless cycle H, the parity of the set of odd edges of H satisfies a given requirement (is even, is odd). Klaus Truemper has proven an important theorem about the minimal forbidden subgraphs for this property. We give a novel and easy proof for this result. We also describe several applications of this theorem: In particular,
-The characterization of Tutte of regular matroids and Reid for matroids representable over GF3
-The characterization of universally signable ghraphs
-The characterization of balanceable graphs
-The characterzation of the graph that can be signed so that every hole is either even or odd.
These results were jointly obtained with Cornuejols, Kapoor and Vuskovic.

Gerard Cornuéjols

Ideal and Balanced Matrices

University of Toulouse and Carnegie Mellon University

A $0, 1$ matrix A is perfect if the polytope $\{x \leq 0 : Ax \leq 1\}$ is an integer polytope. It is ideal if $\{x \geq 0 : Ax \geq 1\}$ is an integer polyhedron.

Jorgen Bang-Jensen

A Polynomial Algorithm For The Hamiltonian Cycle Problem For Semicomplete Multipartite Digraphs

Odense University, Denmark

A *semicomplete digraph* is a digraph whose underlying graph is complete. A *semicomplete multipartite digraph* is a digraph that can be obtained from some semicomplete digraph D by choosing a spanning collection of vertex disjoint induced subgraphs of D and deleting all arcs inside each of these. When the number of subgraphs above is precisely two, we obtain the *semicomplete bipartite digraphs*. While the Hamiltonian cycle problem is very easy for semicomplete digraphs and relatively easy for semicomplete bipartite digraphs (in which case there is still a nice mathematical characterization), it has been an open problem for more than 10 years in the case of semicomplete multipartite digraphs. Recently the authors have proved (constructively) that there is a polynomial algorithm for the Hamiltonian cycle problem for semicomplete multipartite digraphs. In this talk we will give the necessary background and, if time permits, a very brief sketch of the algorithm.

Kathie Cameron

Finding a b-Detachment of a Graph with a Minumum Number of Components

Technical University of Denmark

Let $G = (V, E)$ be a graph and $b = (b(v) : v \in V)$ be a vector of positive integers. A b-detachment of G is a graph obtained from G by splitting each vertex v into $b(v)$ vertices. More formally, a b-detachment of G is obtained by replacing each vertex v by a set $S(v)$ of $b(v)$ vertices, and replacing each edge $e = (u, v)$ of G by an edge with one end in $S(u)$ and the other end in $S(v)$. Nash-Williams

5 András Frank
Orientations of graphs and submodular flows

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ABSTRACT Generalizing an earlier result of H.E. Robbins [1939], C.St.J.A. Nash-Williams [1960] proved that an undirected graph G has a k-edge-connected orientation if and only if G is $2k$-edge-connected. In a recent paper Nash-Williams [1995] found a necessary and sufficient condition for the existence of a strongly-connected orientation of a mixed graph so that every node v has at least a prescribed number of newly oriented edges entering v. It was known earlier how the first of these theorems derives from the theory of submodular flows. In this paper we describe how (a generaliztion of) the second does. As a main device, we prove a simplified feasibility theorem for submodular flows constrained by crossing submodular functions.

I. INTRODUCTION

Let $G = (V, E)$ be an undirected graph and $h : 2\ V \rightarrow \mathbf{Z} \cup \{-\infty\}$ an integer-valued set-function with $h(\emptyset) = h(V) = 0$. The general form

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of the orientation problem we consider consists of finding an orientation of the edges of G so that in the resulting digraph $\vec{G}$ there are at least $h(X)$ edges entering X for every subset $X \subseteq V$. More precisely, the goal is to find necessary and sufficiant conditions for the existence of such an orientation. (Sometimes the problem is formulated in an equivalent form where h is defined only on a family $\mathcal{F}$ of subsets of V and h is finite-valued. In this case h may be extended to each subset of V by defining $h(X) = -\infty$ for subsets not in $\mathcal{F}$.)
We will also consider orientation problems when the input is a mixed graph $M = (V, E + \vec{A})$ composed from an undirected graph $G = (V, E)$ and from a directed graph $\vec{D} = (V, \vec{A})$. In this case the elements of E has to be oriented and $\vec{D}$ will serve only to express the requirement for the orientation.
This problem formulation is too general in the sense that NP-complete problems may be formulated as a special case. Therefore we restrict our attention to a special class of functions, namely, when h is **crossing G-supermodular,** that is,

$$h(A) + h(B) \leq h(A \cup B) + h(A \cap B) + d(A, B) \qquad (1.1)$$

holds for every pair $\{A, B\}$ of subsets of V for which none of $A - B, B - A, A \cap B, V - (A \cup B)$ is empty where $d(A, B)$ denotes the number of edges in G with one and in $A - B$ and the other end in $B - A$.
All orientation problems to be considered in this paper may be described by such a function. There are some other orientation theorems which do not fit this framework, most notably, Nash-Williams' [1960] difficult theorem (which may be called the strong orientation theorem) on the existence of well-balanced orientations. This easily implies Nash-Williams' weak orientation Theorem 1.1 (mentioned in the abstract).

The earliest orientation result is due to H.E. Robbins [1939] who proved that G *has a strongly connected orientation if and only if* G *is 2-edge-connected.*

(A digraph is called **strongly connected** if there is a directed path from every node to every other. More generally, a digraph is called k-**edge-connected** if there are k edge-disjoint paths from every node to

15 Shorter abstracts

Gautam Appa

Bimodular Network Matrices

LSU, London

Network matrices provide an important class of totally unimodular matrices which arise as a natural extension of the node-arc incidence matrices of directed graphs. They can be interpreted as arc-path incidence matrices in which each column represents the unique path between a pair of vertices in a directed tree graph, while rows represent the arcs of the tree graph. We provide a similar extension for the node-edge incidence matrices of undirected graphs. The node-edge incidence matrix (call it N) of an undirected graph, consisting of exactly two +1 entries in each column, has two important properties. First, N is totally bimodular, i.e., any square sub-matrix of N has a determinant 0 or $2k$ where k is any non- negative integer. Second, N is 2-integral. Non-singular submatrices of N contain, in general, decomposable balanced sub-matrices of type $R1$ with determinant $q1$ or type $R2$ with determinant $q2$. We identify the graphical structure of each type. $R1$ represents the incidence matrix of a tree graph augmented by one self-loop, while $R2$ turns out to be the incidence matrix of a connected graph with exactly one odd cycle of length at least three. We concentrate on the only new structure in this setting, viz., incidence matrices of connected graphs with exactly one odd cycle of length three or more. Let G represent a connected graph with n nodes and n edges, having exactly one such odd cycle. Then we define generalised paths/tours involving any two nodes of G, and write a bimodular network matrix, called BINET and denoted by B, as the edge-tour incidence matrix of these generalised tours. In general, elements of B are $0, q1$ and $q2$. We show that a binet matrix B is totally bimodular and 2-integral. We show connections to the ternary packing and covering problems. One surprising result shown is that one of the two five by five unimodular matrices, known not to be a network matrix, is a binet matrix.

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every other node. By Menger's theorem this is equivalent to requiring that every non-empty, proper subset of nodes has at least k entering edges. An undirected graph is called k-**edge-connected** if every cut contains at least k edges.)
Nash-Williams generalized Robbins' theorem, as follows.

THEOREM 1.1 [Nash-Williams, 1960] *For any positive integer k an undirected graph $G = (V, E)$ has a k-edge-connected orientation if and only if G is $2k$-edge-connected.*

F. Boesch and R. Tindell [1980] found another extension of Robbins' theorem concerning orientations of mixed graphs. Let $M = (V, E+\vec{A})$ be a mixed graph. We say that a path P from u to v in M is **correct** if P may use undirected edges arbitrarily and directed edges pointing forward along the path. M is called **traversable** if there is a correct path from u to v for every ordered pair of nodes (u, v). This is easily seen to be equivalent to requiring that M has no directed cuts and the undirected graph arising from M by de-orienting the directed edges is connected. (A **directed cut** of a mixed graph is the set of directed edges entering some $\emptyset \subset X \subset V$ provided that there are no directed edge leaving X and that there are no undirected edges connecting X and $V - X$.) Note that in case M is undirected (that is, $\vec{A} = \emptyset$) traversable is the same as connected, while if M is directed, traversable is the same as srongly connected.

Boesch and Tindell proved: *A mixed graph M has a strongly connected orientation if and only if M is traversable and has no undirected cut edges.*

(A short proof of this result consists of a greedy-type procedure that considers the undirected edges in an arbitrary order and orient them one by one in such a way that no directed cut arises, that is, the traversability is preserved. It can rather easily be shown that, at every step, among the two possible orientations of the current edge at least one will always do).
The question naturally emerges: when does there exist a k-edge-connected orientation of a mixed graph? This can be answered with the

use of submodular flows. The notion of submodular flows was introduced by J. Edmonds and R. Giles [1977]. They proved (among others) that the submodular flow polyhedron is integral. It was observed in [Frank 1982] that there is a strong link between $0-1$-valued submodular flows and orientations of graphs. For example, the integrality of the submodular flow polyhedron easily implies Nash-Williams' result. (It is a great challenge to relate Nash-Williams' strong orientation theorem to submodular flows or more general integral polyhedra.)
The generality of the notion of submodular flows made it possible to derive several extensions of the weak orientation theorem that will be accounted in the next section. One of the most general problem of this type concerns degree-constrained k-edge-connected orientations of mixed graphs. Unfortunately the necessary and sufficient condition is pretty complicated (due to the fact that the corresponding feasibility theorem for submodular flows is complicated.)
Recently, however, Nash-Williams [1995] found a much simpler characterization for the special case $k = 1$ (:strong connectivity). To formulate his result we need some notions and notation. For an undirected graph G, $d_G(X)$ denotes the number of edges between X and $V - X$. Let $e(X) = e_G(X)$ (respectively, $i(X) = i_G(X)$) denote the number of edges with at least one end (with both ends) in X. For a digraph $\vec{D} = (V, \vec{A})$, $\varrho(X) := \varrho_{\vec{A}}(X) := \varrho_{\vec{D}}(X)$ denotes the number of edges entering X and is called the **in-degree** of X (in $\vec{D}$). The function $\varrho_{\vec{D}}$ is called the **in-degree function** of $\vec{D}$. Similarly, $\delta(X) := \delta_{\vec{A}}(X) := \delta_{\vec{D}}(X)$ denotes the number of edges leaving X.
Let $M = (V, E + \vec{A})$ be a mixed graph. It is clearly an equivalence relation of the nodes when two nodes u and v are in relation if there are correct paths from u to v and from v to u. An equivalence class C is called a **di-component.** If no directed edge enters C, it is called an **initial di-component.** Clearly, M is traversable precisely if there is only one equivalence class.
For a subset $Z \subset V$, let $c(M, Z)$ denote the number of those initial di-components C of $M - Z$ which are not entered by any directed edge with tail in Z. Since no undirected edge connects C and $V - (Z \cup C)$, for any strongly connected orientation of M there must be at least one newly oriented edge with head in C and tail in Z, and therefore

Bollobás gave ingenious strategies for the properties of containing a complete graph of order at least k, $2 \leq k \leq n$ as well as for $\chi(G) \geq k$, $2 \leq k \leq n$, where here $\chi(G)$ denotes the chromatic number of G.
By using the topological method, it was shown in [15] that the properties $\Delta(G) \leq k$, $0 \leq k < n-1$ and $\nu(G) \leq k$, $0 \leq k < \lfloor n/2 \rfloor$ are elusive as well. Here, $\Delta(G)$ and $\nu(G)$ denote the maximum degree and the matching number, respectively. Please note that you get more elusive properties by considering $2^T \setminus \mathcal{P}$ and $\{H | H$ is the complementary graph of some graph in $\mathcal{P}\}$.
We close this survey by another open problem:

Problem 4 *Prove that the property of containing a Hamiltonian cycle is elusive.*

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(b) $p > 3$: Similarly to (a), the orbits of edges joining nodes in V_i and V_j, $i \neq j$, are easily seen to contain some complete bipartite graph $K_{p,p}$ which has girth 4 and therefore cannot be contained in $\mathcal{P}$.

The orbits of edges joining two points in V_i are isomorphic to mC_p, the graph consisting of m disjoint copies of the p–cycle C_p, and there are $(p-1)/2$ such orbits. We claim that the union of two such orbits contains a 4–cycle.

It obviously suffices to show that the union of two orbits of G acting on the unordered pairs of $\mathbf{Z}_p$ contains a 4–cycle. These orbits are given by the representatives $\{\{0,j\} | 1 \leq j \leq (p-1)/2\}$. But for each $x \in \mathbf{Z}_p$, the vertices $x, x+j, x+j+k, x+k$ lie on a 4–cycle in the union of the orbits containing $\{0,j\}$ and $\{0,k\}$.

It follows that $\chi(\mathcal{P}_{G \wr H}) = (p-1)/2 > 1$, again a contradiction.

□

The last two theorems suggest the following problem:

Problem 2 *Prove that all decreasing graph properties consisting of triangle–free graphs only are elusive.*

14.4 Special Graph Properties.

In this last section, we list some of the results on the elusiveness of graph properties often considered by graph theorists. A natural approach is to find a strategy for player $\mathcal{S}$ and show that this strategy forces $\mathcal{A}$ to probe all possible edges. It is not difficult to characterize all graph properties for which a so–called *simple strategy* works:
Say either always "yes" or always "no". (see [1, 3, 13, 14]).
By a simple strategy, we can, e.g., prove the elusiveness of *planarity* or *connectedness*. For 2–connectedness, a more refined strategy works (see [13]). We offer the following problem:

Problem 3 *Prove that k–connectedness is elusive for $k \geq 3$.*

there are at least $c(M,Z)$ newly oriented edges leaving Z.
Let $f : V \to \mathbf{Z}$ be a non-negative, integer-valued function and let $f(Z) := \sum_{v \in Z} f(v)$.

THEOREM 1.2 [Nash-Williams, 1995] *A mixed graph $M = (V, E + \vec{A})$ has a strongly connected orientation $(V, \vec{E} + \vec{A})$ satisfying*

$$\varrho_{\vec{G}}(v) \geq f(v) \text{ for every } v \in V \tag{1.2}$$

if and only if M is traversable, M has no undirected cut-edge, and

$$e_G(Z) \geq f(Z) + c(M,Z) \tag{1.3}$$

holds for every non-empty subset $Z \subseteq V$.

From the above considerations the necessity of (1.3) is easy. Indeed, for an orientation $(\vec{G} + \vec{D})$ of M satisfying the requirements, one has $e_G(Z) = \sum_{v \in Z} \varrho_{\vec{G}}(v) + \delta_{\vec{G}}(Z) \geq f(Z) + c(M,Z)$.
Actually, Nash-Williams considered this orientation problem in a slightly different form. He wanted to find a partial orientation of the undirected edges of M (that is, not necessarily all undirected edges have to get oriented) so that the resulting mixed graph is traversable and $(*)$ there are exactly $f(v)$ newly oriented edges entering every node v. This problem, however, is equivalent to the one in Theorem 1.2. Indeed, if a required partial orientation exists, then, by the theorem of Boesch and Tindell, the remaining undirected edges can be oriented so as to obtain a strongly connected digraph. This orientation clearly satisfies (1.2). Conversely, if there is a strongly connected orientation satisfying (1.2), then de-orienting a sufficient number of newly oriented edges we obtain a traversable mixed graph in which $(*)$ holds.

In the present paper we will show how a generalization of Theorem 1.2 can be derived via submodular flows. As a main tool, we prove a simplified feasibility theorem for submodular flows constrained by crossing submodular functions. The simplification is based on a the notion of full-truncation of a set-function, a new form of what was called earlier bi-truncation.

6 Anna Galluccio and Martin Loebl Even cycle in directed graphs

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Abstract

The aim of this note is to survey different concepts together with the results obtained so far, related to the **Even Cycle Problem**, introduced in the beginning of seventies by Younger. This seemingly innocent problem to decide whether a directed graph contains an even length directed cycle belongs to the central problems of combinatorial optimization.

6.1 Permanents and Determinants

Let $A = (a_{ij})$ be an $n \times n$ matrix and π a permutation in the symmetric group S_n. The *permanent* of A is defined as:

$$perm(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i\pi(i)}$$

and it differs from computing the determinant of A only for the sign of the permutations. In fact,

$$det(A) = \sum_{\pi \in S_n} sgn(\pi) \prod_{i=1}^{n} a_{i\pi(i)}.$$

where $sgn(\pi)$ is $+1$ if the permutation π is even and -1 otherwise. Now, while the determinant of a matrix may be computed in polynomial time using the Gaussian elimination, a computation of the

Recall that the *girth* of a graph is the length of its shortest cycle (the girth of a forest being defined as ∞). We want to prove the following result from [15]:

Theorem 6 *Suppose $\mathcal{P}$ is a decreasing, (nontrivial) graph property such that all graphs of $\mathcal{P}$ have girth greater than 4. Then $\mathcal{P}$ is elusive.*

Proof: If n is a power of 2, then elusiveness follows from Theorem 3. Hence, we may assume that some odd prime p divides n, $n = pm$ say, and $V = \mathbf{Z}_p \times \mathbf{Z}_m$. Let $G := \langle \tau \rangle$, the group generated by $\tau : \mathbf{Z}_p \to \mathbf{Z}_p : x \to x + 1$, and $H := \langle \sigma \rangle$ where $\sigma : \mathbf{Z}_m \to \mathbf{Z}_m : y \to y + 1$. Recall that the *wreath product* $G \wr H$ is a permutation group on V defined as follows:

$$G \wr H := \{(f;\pi) | f : \mathbf{Z}_m \to G, \pi \in H\}$$

For $(i, j) \in V$ and $(f;\pi) \in G \wr H$ we have

$$(f;\pi)(i, j) := (f(\pi(j))(i), \pi(j)).$$

$G \wr H$ contains the normal subgroup

$$G_1 := \{(f; 1_H) | f : \mathbf{Z}_m \to G\}$$

which is isomorphic to the m–fold direct product $G^m = G \times \cdots \times G$. Hence, G_1 is a p–group and we have $(G \wr H)/G_1 \simeq H$. It follows that $G \wr H$ is in $\mathcal{G}$.
Now suppose there was a nonelusive decreasing graph property $\mathcal{P}$ which contained graphs of girth greater than 4 only. By Theorems 1 and 2 we must have

$$\chi(\mathcal{P}_{G \wr H}) = 1.$$

To obtain a contradiction, we consider two cases:

(a) $p = 3$: The orbit of an edge joining two vertices in some set $V_i := \mathbf{Z}_p \times \{i\}$, $i \in \mathbf{Z}_m$, is isomorphic to mK_3, the graph consisting of m disjoint triangles. But K_3 has girth 3. The orbit of an edge joining vertices in V_i and V_j $(i \neq j)$ is easily seen to contain the complete bipartite graph $K_{3,3}$ as a subgraph which has girth 4. We infer that $\mathcal{P}_G$ is empty, a contradiction.

Problem 1 *Prove that all nontrivial, monotone graph properties satisfy*

$$c(\mathcal{P}) \geq \binom{n}{2} + o(n^2).$$

The topological method has been used afterwards by several people. V. King applied it to digraph properties and A. Yao proved the following interesting result:

Theorem 4 (Yao) *All nontrivial, monotone bipartite graph properties are elusive.*

Here, a bipartite graph property means the following: Fix two disjoint, nonempty and finite sets U and W and let $T := \{uw : u \in U,\ w \in W\}$. A bipartite graph property is a subset of 2^T which is invariant under all permutations of U and W.
Proof: It suffices to consider a cyclic group G acting transitively on one colour class, say W. Let some decreasing bipartite graph property $\mathcal{P}$ be given. Then $\mathcal{P}_G$ is a complex on the set of all stars with centers in U. By isomorphism invariance and monotonicity, there is some number $k \leq u := |U|$ such that $\mathcal{P}_G$ consists of all sets of at most k stars. If $\mathcal{P}$ is nontrivial, $k < u$ and we have:

$$1 - \chi(\mathcal{P}_G) = \sum_{i=0}^{k} (-1)^i \binom{u}{i}$$

$$= \sum_{i=0}^{k} (-1)^i \left[\binom{u-1}{i-1} + \binom{u-1}{i} \right]$$

$$= (-1)^k \binom{u-1}{k} \neq 0.$$

It follows that $\mathcal{P}$ is elusive.

□

Recently, the following analogue of Yao's theorem was proved (see [16]):

Theorem 5 *All (nontrivial) decreasing graph properties consisting of bipartite graphs only are elusive.*

permanent seems to be much harder. Valiant ([27]) proved that it is $\#P$-complete, even for $(0,1)$-matrices.
In 1913, Polya ([18]) suggested computing the permanent of a $(0,1)$-matrix by multiplying some entries by -1 so that the original permanent becomes the determinant of the resulting matrix. Such a transformation however needs not exist, as noticed in the same year by Szego ([23]).
Following this scheme let us call a $(0,1,-1)$-matrix *rigid* if the nonzero terms in its determinant expansion all have the same sign. A $(0,1)$-matrix is called *convertible* if it is possible to multiply some of its entries by -1 to get a rigid matrix.
Note that the permanent of a $(0,1)$-matrix which is the adjacency matrix of a bipartite graph G equals to the number of perfect matchings of G.

6.2 The Number of Perfect Matchings and Pfaffians

Let D be a digraph on n vertices. Let $B(D) = (b_{ij})$ be a skew-symmetric $m \times m$ matrix defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i v_j \text{ is an arc of } D \\ -1 & \text{if } v_j v_i \text{ is an arc of } D \\ 0 & \text{otherwise} \end{cases}$$

The *Pfaffian* of a skew-symmetric matrix B of size $2n \times 2n$ is defined as

$$Pf(B) = \sum_{P} sgn(P) \times b_{i_1 j_1} \cdots b_{i_n j_n}$$

where $P = \{\{i_1 j_1\}, \cdots, \{i_n j_n\}\}$ is a partition of $\{1, \ldots, 2n\}$ into pairs and $sgn(P)$ equals to the sign of the permutation $(i_1 j_1, \ldots, i_n j_n)$.
By a classical result of Cayley $det(B) = Pf^2(B)$ for a skew-symmetric matrix B.
Each nonzero term of the expansion of the Pfaffian of $B(D)$ corresponds to a perfect matching of D. Hence all the nonzero terms of the expansion of the Pfaffian of $B(D)$ have the same sign if and only if $M(D)^2 = det(B(D))$, where $M(D)$ denotes the number of perfect

matchings of the underlying graph of D. A digraph is called *Pfaffian* if it has this property. A digraph obtained by orienting each edge of a graph G is called an *orientation* of G. A graph is *Pfaffian* if it has a Pfaffian orientation.
Summarizing, if G is Pfaffian and we know its Pfaffian orientation $D(G)$, the number of perfect matchings of G may be computed in polynomial time since its square is equal to $det(B(D(G)))$.
This method has been proposed by Kasteleyn in the beginning of sixties in order to compute the number of perfect matchings of general graphs ([12]). He also showed that all planar graphs are Pfaffian and proved that a digraph D is Pfaffian if and only if each alternating cycle C of the underlying undirected graph G (with respect to an arbitrary perfect matching of G) has an odd number of edges directed clockwise. For an extension of the Kasteleyn's method see ([8]).
In 1975 Little [15] observed that a matrix is convertible if and only if it is the incidence matrix of a Pfaffian bipartite graph. In this seminal paper he also characterized the Pfaffian bipartite graphs in terms of a forbidden structure: he showed that a bipartite graph G is Pfaffian if and only if it has an even subdivision K of $K_{3,3}$ with a perfect matching in $G - V(K)$.

6.3 Minimally non-bipartite hypergraphs, sign-nonsingular matrices and even digraphs

A *hypergraph* is a set of finite subsets (called *edges*) of a set whose elements are called *vertices*. A hypergraph is *bipartite* if there is a subset of vertices intersecting all the edges and containing none completely.
In 1973 Seymour ([21]) proved that the recognition of non-bipartite hypergraphs with the same number of edges as vertices is equivalent to the even cycle problem. This is important since the recognition of general bipartite hypergraphs is an NP-complete problem.
A *sign-nonsingular matrix* is a real matrix such that the columns of any real matrix of the same size and with the same sign pattern are linearly independent.
This notion appears in economics models, for details see the paper of Lancaster ([14]).

Denote by $\mathcal{G}$ the set of all finite groups G containing a normal p–subgroup G_1, $G_1 \trianglelefteq G$, such that the factor group G/G_1 is cyclic. If G is a permutation group on T leaving $\mathcal{P}$ invariant, we define the simplicial complex $\mathcal{P}_G$ with vertex set $T_G := \{B \in \mathcal{P} : B \text{ orbit of } G\}$ by

$$\{B_1, \ldots, B_k\} \in \mathcal{P}_G :\Longleftrightarrow B_1 \cup \ldots \cup B_k \in \mathcal{P}.$$

The following result of P.A. Smith and R.G. Oliver (see [12] and [9]) is crucial:

Theorem 2 (Smith, Oliver) *If $G \in \mathcal{G}$ is a permutation group on T leaving the $\mathbf{Z}_p$–acyclic complex $\mathcal{P}$ invariant, then*

$$\chi(\mathcal{P}_G) = 1.$$

In particular, $\mathcal{P}_G$ is nonempty.

14.3 Some applications.

In case that the number of vertices is a prime power, Kahn, Saks and Sturtevant assumed that the vertex set V is the Galois field $GF(n)$ and considered the group G of affine–linear transformations "$\tau(x) = ax + b$" with $a, b \in V$ and $a \neq 0$. Clearly, G contains the normal subgroup of all translations ($a = 1$) which is isomorphic to the additive group of the field V and thus is a p–group. The factor group G/G_1 is isomorphic to the multiplicative group of the field V which is cyclic. It follows that G is in $\mathcal{G}$. Any (nontrivial) decreasing graph property is clearly invariant under G. If it is not elusive, then, by the above result, the complex $\mathcal{P}_G$ is nonempty. But G acts doubly transitive on V and thus transitive on the set $V^{(2)}$. We conclude that $\mathcal{P}$ contains the complete graph on V and thus all graphs on V contradicting our assumption that $\mathcal{P}$ is nontrivial. This is the proof of

Theorem 3 (Kahn, Saks and Sturtevant) *If $n = |V|$ is a prime power, then all nontrivial, monotone graph properties are elusive.*

It is a pity that nobody knows how to get an $(n^2/2) + o(n^2)$–bound from this for general n. Since a proof or disproof of Karp's Conjecture can be very difficult, we separately state the following problem:

in [14].
Then, in 1984, Kahn, Saks and Sturtevant [8] proved that $c(\mathcal{P}) \geq (n^2/4) + o(n^2)$. This is up to now the best approximation for the following conjecture of Karp:
All nontrivial, monotonic graph properties are elusive.
In fact, Kahn, Saks and Sturtevant proved Karp's conjecture in case n is a prime power (see below). In the following sections, we are now going to describe their method (which uses algebraic topology) and show several applications.

14.2 Tools from algebraic topology.

From now on, we assume that $\mathcal{P}$ is a monotone, nontrivial graph property.
Since $c(\mathcal{P}) = c(2^T \setminus \mathcal{P})$, it suffices to study decreasing graph properties. A decreasing graph property can be seen as an (abstract) simplicial complex. Hence, its *Euler characteristic* $\chi(\mathcal{P})$ is defined as

$$\chi(\mathcal{P}) := \sum_{i=1}^{|T|} (-1)^{i-1} a_i = 1 - q(\mathcal{P}, -1)$$

If G is an abelian group (here: $G = \mathbf{Z}$ or $G = \mathbf{Z}/p\mathbf{Z} = \mathbf{Z}_p$, p prime), the complex is called *G–acyclic* if

$$H_0(\mathcal{P}, G) = 0 \quad \text{and} \quad H_i(\mathcal{P}, G) = 0 \quad \text{for } i > 0$$

Here, $H_i(\mathcal{P}, G)$ denotes the i–dimensional homology group of $\mathcal{P}$ with respect to G. Kahn, Saks and Sturtevant established the following connection between complexity theory and algebraic topology :

Theorem 1 (Kahn, Saks and Sturtevant) *If $\mathcal{P}$ is not elusive, then it is contractible and hence $\mathbf{Z}_p$–acyclic for all primes p. In particular, $\chi(\mathcal{P}) = 1$.*

The proof of this result is quite easy and similar to the proof of divisibility of $q(\mathcal{P}, z)$ by $1 + z$ if $\mathcal{P}$ is not elusive.
It is a striking fact discovered by P.A. Smith, that in some cases the condition $\chi(\mathcal{P}) = 1$ is preserved when the complex $\mathcal{P}$ is replaced by a much simpler "complex of orbits".

The problem of recognizing sign-nonsingular matrices is proved to be NP-complete by Klee, Ladner and Manber ([13]) and also Brualdi and Shader ([3]). However, the important special case of square sign-nonsingular matrices is open, and in fact it is simply equivalent to the rigidity of matrices.
It is proved by Butkovic ([4]) that the even cycle problem is equivalent to testing regularity of matrices in min-algebra.
In 1987 Seymour and Thomassen [22] introduced *even digraphs*. A digraph D is *even* if any subdivision of D has an even directed cycle. They showed that the Even Cycle Problem is equivalent to the Even Digraphs Recognition Problem and they characterized even digraphs in terms of a forbidden structure. This has been extended in [10] to a characterization of the digraphs D such that each subdivision of D contains a directed cycle of length different from p modulo q, for any p, q.
In 1989 Vazirani and Yannakakis [28] observed that the Even Digraphs Recognition Problem is equivalent to recognition of convertible matrices. Hence recognition of even digraphs and of Pfaffian bipartite graphs are computationally equivalent. Moreover, the characterization of even digraphs due to Seymour and Thomassen and the characterization of Pfaffian bipartite graphs due to Little may be deduced one from the other (see [11]).

6.4 Computational Results

In 1975 Lovász ([17]) raised two questions:
Does there exist k such that any digraph in which there are at least k arcs leaving each vertex has a directed cycle of an even length?
Does there exist k such that strongly k-connected digraph has a directed cycle of an even length?
Partial results on these two questions were given by Friedland ([7]) and Alon and Linial ([1]). Thomassen disproved the first question in [24] and settled the second one affirmatively in [25]. He proved that each strongly 3-connected digraph has a directed cycle of an even length. Each such digraph must have a directed cycle of length different from p modulo q for any p, q as observed in [10].
It is proved by Chung, Goddard and Kleitman ([5]) that every strongly connected digraph with n vertices and at least $(n+1)^2/4$

arcs must contain a directed cycle of even length.
In 1990 a polynomial-time algorithm to solve the Even Cycle Problem in the class of planar digraphs was proposed by Thomassen ([26]) and independently in [9].
This was extended in [11] by showing that the Even Cycle Problem (and more general modularity problems as well) for the digraphs of any class $\mathcal{D}$ closed for taking subdigraphs and subdivisions is solvable in polynomial time in case the P_i-homeomorphism Problem, $i \leq 5$, may be solved in polynomial time in $\mathcal{D}$. P_i denotes a path with i vertices. The *H-homeomorphism Problem* is defined as follows:

> let $H = (V(H), A(H))$ be a fixed digraph. Given a digraph $D = (V(D), A(D))$ and a function f from $V(H)$ to $V(D)$ such that $f(x) \neq f(y)$ for $x \neq y$, decide whether there is a homeomorphism from H to D which extends f.

The H-homeomorphism problem is known to be polynomially solvable in the class of acyclic digraphs (i.e. digraphs without directed cycles) (see [6]) and in any class of digraphs embedded on a fixed surface (see [20, 19]). Hence, the same holds for the Even Cycle Problem.
On the other hand, if H is not a star where all arcs go from its center or to its center then the H-homeomorphism problem in the class of all digraphs is NP-complete (see [6]).
In the end of 1996, Robertson, Seymour and Thomas announced that the Even Cycle Problem is polynomially solvable for general digraphs.
In the end, let us mention that the problem "Does a digraph D have a directed cycle of length p modulo q" is proved to be NP-complete for $p > 0$ and $q > 2$ in [2].

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Hence, if $(1+z)^k$ does not divide $q(\mathcal{P}, z)$, it does not divide at least one of the summands $q(\mathcal{P}_i, z)$. Now $\mathcal{S}$ answers in such a way that the set of all sets in $\mathcal{P}$ compatible with all the answers given has an enumeration polynomial not divisible by $(1+z)^k$. Following the same strategy for questions $2, 3, \ldots$, we see that after $|T| - k$ questions, the subsets of T compatible with all the answers form an interval $\mathcal{I}$ of length k in the Boolean lattice 2^T. Because $(1+z)^k$ does not divide the enumeration polynomial of $\mathcal{I} \cap \mathcal{P}$, it follows that some of the sets in $\mathcal{I}$ are in $\mathcal{P}$ and some are not. Hence, at least one further question must be asked. In particular, if $1 + z$ does not divide $q(\mathcal{P}, z)$, then $\mathcal{P}$ is elusive.
The recognition complexity has been studied by various authors during the last 23 years (see all the references except [9] and [12]). The greatest efforts have been devoted to the case that T equals $V^{(2)}$, the set of two–element subsets of some finite set V. In this case the subsets of T can be interpreted as graphs with vertex set V. A *graph property* is a subset $\mathcal{P} \subset 2^T$, $T = V^{(2)}$, such that $\mathcal{P}$ contains with each graph G also each isomorphic copy of G (with vertex set V). A graph property $\mathcal{P}$ is called *nontrivial* if $\mathcal{P} \neq \emptyset, \mathcal{P} \neq 2^T$. It is called *decreasing* if, for each $G \in \mathcal{P}$, all subgraphs of G (with vertex set V) are contained in $\mathcal{P}$, *increasing* if $2^T \setminus \mathcal{P}$ is decreasing and *monotonic* if it is increasing or decreasing.
In 1973, Rosenberg [10] conjectured that there exists some $\gamma > 0$ such that

$$c(\mathcal{P}) \geq \gamma n^2$$

for all nontrivial graph properties $\mathcal{P}$, $n = |V|$.
This conjecture was soon disproved (see [2]) and, together with Aanderaa, Rosenberg modified his conjecture as follows:
There exists $\gamma > 0$ such that for all nontrivial, monotonic graph properties $\mathcal{P}$ we have $c(\mathcal{P}) \geq \gamma n^2$.
This version turned out to be true and was proved by Rivest and Vuillemin in 1975 with $\gamma = 1/16$ (see [11]) by first proving that $c(\mathcal{P}) \geq n^2/4$ if n is a power of two and then applying some interpolation argument for the other values of n. Kleitman and Kwiatkowski [7] improved the value of γ from $1/16$ to $1/9$ for n large. They proceeded in a way similar to Rivest–Vuillemin but showed that $c(\mathcal{P}) \geq n^2/3$ if n is a power of three. Both results were proved by a different method

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14.1 Introduction.

Let T be some finite set and denote by $\mathcal{P}$ a property of subsets of T, i.e. $\mathcal{P} \subset 2^T$, the power set of T. Imagine two players $\mathcal{A}$ (*Algy*) and $\mathcal{S}$ (*Strategist*). playing the following game: Player $\mathcal{A}$ wants to learn from $\mathcal{S}$ whether an unknown set $X \subset T$ is in $\mathcal{P}$ or not by asking him questions of the form *Is* $x \in X$? ($x \in T$). The goal of $\mathcal{A}$ is to minimize the number of questions but $\mathcal{S}$ wants to force $\mathcal{A}$ to ask as many questions as possible. If both players play optimally from their point of view, then the number of questions which are asked in the game is called the *recognition complexity* of $\mathcal{P}$ (also: *Boolean decision tree complexity*) and is denoted by $c(\mathcal{P})$. If $\mathcal{S}$ can force $\mathcal{A}$ to probe all elements of T, then $\mathcal{P}$ is called *elusive* (also: *evasive*). As an example for proving lower bounds for $c(\mathcal{P})$, associate with $\mathcal{P}$ its so-called *enumeration polynomial* $q(\mathcal{P}, z)$ (first introduced in [2] and [11]):

$$q(\mathcal{P}, z) := \sum_{X \in \mathcal{P}} z^{|X|} = \sum_{i=1}^{|T|} a_i(\mathcal{P}) z^i.$$

Then $a_i(\mathcal{P})$ is the number of i–element sets in $\mathcal{P}$. If $(1+z)^k$ does not divide $q(\mathcal{P}, z)$, then $c(\mathcal{P}) > |T| - k$. In fact, suppose that $\mathcal{A}$ asks the element x as his first question. This question divides $\mathcal{P}$ into two disjoint subsets $\mathcal{P}_1$ and $\mathcal{P}_2$, namely the sets in $\mathcal{P}$ containing x and those which do not. We clearly have

$$q(\mathcal{P}, z) = q(\mathcal{P}_1, z) + q(\mathcal{P}_2, z).$$

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problem, but the NP-completeness of the triangulation problem does not seem to have immediate consequenses to the dual problem. First of all, we do not know the dual graph before we know the embedding. Secondly, the property of being a minimum genus embedding is not preserved under duality.
Also, the complexity of a graph problem may change when we restrict the problem to cubic graphs. For example, the graph isomorphism problem for cubic graphs is polynomial. But, the complexity for general graphs is not known.
In a forthcoming paper in *J.Combinatorial Theory B*, I prove the following.

Theorem. The genus problem for cubic graphs in NP-complete.

The proof easily extends to r-regular graphs for r equal to 4 or 5 and probably, with a little more effort, to any fixed r. But, if r is not fixed the situation is different. I believe that the genus problem is polynomial for graphs with n vertices all of which have degree $99n/100$ since I believe that every such graph (almost) triangulates a surface in which case the genus is simply determined by the number of vertices of the graph. The complexity may be different for graphs with n vertices all of which have degree $n/100$.

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The genus problem for graphs

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The genus problem, which is mentioned as unsolved problem 3 in the book by Garey and Johnson on NP-completeness is the following: Given a graph G and a nonnegative integer g, is it possible to draw (or embed) G on the orientable surface of genus g, that is the sphere with g handles added. A special variant was suggested already at the 1976 Kalamazoo conference by Gerhard Ringel: Given a graph G, is it possible that G triangulates some orientable surface. If g denotes the genus of that surface, and n, e denote the number of vertices and edges, respectively, of G, then Euler's formula implies that

$$e = 3n - 6 + 6g.$$

Hence there is only one orientable surface that G may triangulate. Any other surface that G can be drawn on must have larger genus. Hence Ringel's question is a special case of the genus problem. The motivation for Ringel's question is that a sufficiently simple solution the the triangulation problem might result in a unified proof of the genus problem for complete graphs which was raised in 1890 by Heawood and solved about 80 later by Ringel and Youngs.
In 1989 (*J.Algorithms* **10**, 568-576) I proved that the genus problem is NP- complete. In 1993 (*J.Combinatorial Theory B* **57**, 196-206) I proved that even the more special problem of Ringel is NP-complete. In 1991 Bruce Richter (private communication) pointed out that vertices of large degree play an important role in the construction and asked about the complexity of the genus problem when restricted to cubic graphs. In some sense, this problem is dual to the triangulation

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Disconnected vertex sets and equidistant code pairs

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Let G be a graph with vertex set V. Two disjoint subsets A and B of V are *disconnected* if there is no edge between A and B. We define $\Phi(G)$ to be the maximum of $\sqrt{|A| \cdot |B|}$ for disconnected sets A and B in G. Suppose V is the set of words of length n over an alphabet $\{1, \ldots, q\}$ and define two words adjacent if their Hamming distance (i.e. the number of coordinates in which they differ) is not equal to a fixed $\delta \in \{1, \ldots, n\}$. Then a pair of disconnected sets becomes an *equidistant code pair.*
The quantity $\Phi(G)$ has an application in information theory and leads to a lower bound for the two-way communication complexity of functions defined on $V \times V$ that are constant over the non-edges of G. About ten years ago this application caused some activity in the study of equidistant code pairs. The best result is due to Ahlswede [1], who gives the exact value of $\Phi(G)$ for $q = 2$, 4 and 5, for every δ and n.
In this paper we will give a bound for $\Phi(G)$ in terms of the eigenvalues of a matrix associated with G. In case the complement of G is given by a relation of an association scheme the bound takes an easy form, which applied to the Hamming scheme leads to a bound for equidistant code pairs. This bound is not as accurate as Ahlswede's result, but it is more general and it turns out to be sharp for some values of q, n and δ, and for $q \to \infty$ for any fixed n and δ.

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suffices to prove it for digraphs in which each vertex has indegree and outdegree 2, and with a number of other properties. Then we prove a second lemma that says roughly that if G has as a subdigraph a kind of grid together with some additional paths, then $\nu(G)$ is large. Third, we prove that digraphs with the properties implied by the first lemma do contain such grids. Finally, we use (3) to obtain a polynomial-time algorithm for the problem.

A well-known theorem of Erdős and Pósa states the following.

(1) *For any integer $n \geq 0$ there exists an integer $t \geq 0$ such that for every graph G, either G has n circuits that are pairwise vertex-disjoint, or there exists $T \subseteq V(G)$ with $|T| \leq t$ such that T meets every circuit of G.*

In 1973, Younger conjectured that there was an analogue of (1) for directed circuits in digraphs. (The question probably occurred to other workers before then - for instance, it seems that Fulkerson worked on it in the 1950's, but put nothing in print.) For $n = 0$ and 1 this is obvious. For $n = 2$ it was conjectured independently by Gallai (who also considered the $n > 2$ case without publishing it), and eventually was answered by McCuaig in the following strong form.

(2) *For every digraph G, either G has two vertex-disjoint directed circuits, or there exists $T \subseteq V(G)$ with $|T| \leq 3$ such that T meets every directed circuit of G.*
But Younger's conjecture in general (and even for $n = 3$) has remained open until the present. For a digraph G we denote by $\nu(G)$ the maximum n such that G has n directed circuits, pairwise vertex-disjoint, and by $\tau(G)$ the minimum t such that there exists $T \subseteq V(G)$ with $|T| = t$ meeting all directed circuits. Evidently $\nu(G) \leq \tau(G)$, and Younger's conjecture is that $\tau(G)$ is at most $f(\nu(G))$ for some function f independent of G. Thus, our main result is the following.

(3) *For every integer $n \geq 0$ there exists a (minimum) integer $t_n \geq 0$ such that for every digraph G, either $\nu(G) \geq n$ or $\tau(G) \leq t_n$.*
Let t_n be defined as in (3); we prove by induction on n that t_n exists for all n. Obviously $t_0 = t_1 = 0$, and from McCuaig's theorem $t_2 = 3$. We prove an upper bound on t_n, but it is very large (a multiply iterated exponential, where the number of iterations is also a multiply iterated exponential) and probably far from best possible. The best *lower* bound on t_n that we know is $O(nlog(n))$, due to Alon (unpublished).
The proof is organized as follows. First, we apply Ramsey's theorem to deduce a lemma which implies that to prove (3) in general, it

8 Winfried Hochstättler and Jaroslav Nešetřil Farkas' Lemma and Morphism Duality

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There is a great variety of combinatorial optimization results which are of min-max type or "good characterizations" [6] and from the point of view of theoretical computer science these results establish the membership in the class $NP \cap co\text{-}NP$. However, it seems to be convenient for a better understanding to have a more uniform and perhaps technically more fitting description of certain classes of characterizations. The most important here is of course the linear programming approach and the duality of linear programming serves as a prototype and master theorem which implies many (but not all) min-max results. Yet, there are other approaches for example the lattice method of A. Hoffman [16], [17], or general methods related to the Ellipsoid Method [10]; see also [4]. Another possible approach of algebraic flavour was suggested in [22], [18], [12] and relies on the definition of classes by means of special morphisms. (Although this has a definitive category theory flavour one does not make use of any of the deep results of the theory of categories and thus we do not have to use the formal language of this theory.) One proceeds as follows: Suppose that $\mathcal{K}$ is a class of objects together with a certain class of maps ("homomorphisms", "morphisms") between them. Given two objects A, B of $\mathcal{K}$ we denote by $A \to B$ the existence of a map from A to B. Given a fixed object A we denote by

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$$\to (A)$$

the class of objects of $\mathcal{K}$ which map to A:

$$\to (A) \quad := \quad \{B \in \mathcal{K} \mid B \to A\}.$$

Similarly, $(A) \to \quad := \quad \{B \in \mathcal{K} \mid A \to B\}$.
The complementary classes will be denoted by $\not\to (A)$ and $(A) \not\to$. Explicitly, $\not\to (A)$ is the class of objects which do not map into A.
A *homomorphism duality theorem for* $\mathcal{K}$ is the following equation of two classes

$$(B) \to \quad = \quad \not\to (A). \qquad (*)$$

The equation $(*)$ is the equation of two classes and thus, explicitly, it means:
For every object C of $\mathcal{K}$: $\quad B \to C$ iff $C \not\to A$.
Examples of homomorphism duality are numerous. The class of graphs and homomorphisms between them was studied in greater detail in [22], [18], [12], [13] (a homomorphism $f : G \to H$ is any edge preserving mapping of the vertices).
The purpose of the full version of this paper [15] is to demonstrate the power of Homomorphism Duality-scheme $(*)$ by showing that the duality theorem of linear programming, Farkas' Lemma, may be interpreted within this framework. We can show that the standard class of morphisms for oriented matroids yields a particular instance of duality which is equivalent to Farkas' Lemma. Moreover, we show that with properly defined morphisms this is the only existing duality. Thus, in this setting, Farkas' Lemma is the only valid duality (i.e. it is the only valid duality for oriented matroids with a given class of maps.).
The use of oriented matroids for linear programming in the context of homomorphism dualities was suggested by L. Lovász and A. Schrijver [20]. Our approach here is motivated by this and by their question whether there exists a homomorphism duality for Farkas' Lemma of such a form, that on both sides of the equation $(*)$ is the same type of morphisms (as is the case for homomorphisms of graphs; indeed our formulation of a homomorphism duality covers only such cases).

12 Bruce Reed, Neil Robertson, Paul Seymour and Robin Thomas Packing directed circuits

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Thus our main results have the simple form of a class-equation

Theorem A:

$$(\mathcal{L}, 1) \rightsquigarrow \quad = \quad \not\rightsquigarrow (\mathcal{L}^*, 1).$$

where the objects are affine oriented matroids and the morphisms are generalized affine strong maps which we define as alternating chains of strong maps (for our purposes best done by means of the notion of convexity) and their duals. Note that our construction of these maps is analogous to several constructions in homology theory and as well to the definition of a minor by means of homomorphisms.
It is interesting to note that in this setting Farkas' Lemma is the only homomorphism duality. More precisely we have

Theorem B: Let $(B, 1) \rightsquigarrow \quad = \quad \not\rightsquigarrow (A, 1)$ be a homomorphism duality theorem for oriented matroids and generalized affine strong maps. Then $(B, 1)$ is morphic to the loop and $(A, 1)$ is morphic to the coloop.

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It is well-known that J_B is a jump system (see a lemma in [3]), and it follows easily that the faces of J (that is sets of the form $J \cap F$, where F is a face of conv(J)) are also jump systems. It was conjectured by Tamir that in case $a \in \{-1,1\}^n$ the face of vectors in J optimizing a also constitute a jump system. We prove this conjectre in the following sharper form:
Denote $\max(J,a) := \{v \in J : a^T v = \max_{x \in J} a^T x$. The projection (restriction) of a set of vectors $W \subseteq \mathbb{R}^n$ onto $I \subseteq \{1,\dots,n\}$ is $\{(x_i : i \in I) : x \in W\}$.

Corollary 3: *Let J be a jump-system, and $a \in \{-1,0,1\}^n$. The projection F of max(J,a) onto the support of a is a convex jump system.*

Proof: We know that F is a jump system. Suppose without loss of generality that $0 \in \mathbb{R}^n$ is a hole of F. According to Corollary 1 there exists a vector $y \in F_0$ such that $-y \in F_0$, without loss of generality $y \geq 0$. But since F is a jump system, if $y' = y - e_i$ is at one step from y towards $-y$, there exists a second step $y'' = y' - e_j \in F$ at one step from y' towards $-y$. Note that $y' \in F$ is not possible, because then $d(y',0) < d(y,0)$ contradicting $y \in F_0$, and for the same reason $j = i$. Since i is in the support of a, for y and $y'' = y - 2e_i$ (or more precisely for the vectors whose projections are y and y'') $a^T y = a^T y''$ cannot hold. □

The following theorem was first proved by Cunningham many years ago with submodular and matrix techniques.

Corollary 4 *Let J_1 and J_2 be jump-systems. Then conv$(J_1)\cap$conv(J_2) is a half-integer polyhedron.*

Proof: It is enough to prove that conv$(J_1)\cap$conv(J_2), if non-empty, has a half integer point, because then this statement can be applied to faces (which, as already mentioned, are also jump systems). But conv$J_1\cap$conv$J_2 \neq \emptyset$ if and only if $0 \in$conv$\{v_1 - v_2 : v_1 \in J_1, v_2 \in J_2\}$. The set of differences, as it has already been mentioned, is a jump system ([1]), denote it by J. If $0 \in J$, we are done; if not, 0 is a hole, and applying Corollary 1, $0 = x + y$: $x, y \in J$, that is, $x = x_1 - x_2$, $y = y_1 - y_2$, $(x_i, y_i \in J_i, i = 1,2)$. Substituting and rearranging: $x_1 + y_1 = x_2 + y_2$. Dividing by two the left hand side is in conv(J_1), the right hand side in conv(J_2). □

On the example of matchings: a first reason for the nonexistence of a perfect matching in a graph can be that the all 1 vector is not in the convex hull of degree sequences of subgraphs of G. (This is the case if and only if there is no 2-matching in the graph, and then the corresponding violated linear condition is a good certificate.) In case the all 1 vector is in the convex hull the essential part of Tutte's theorem enters the game.
In case 0 is a gap there exists vectors in J_0 which have more particular properties than those claimed by Lovász's theorem:

Theorem: *If J is a jump-system and $0 \in \mathbb{Z}^n$ is a hole in J, then there exists $v \in J_0$ so that $v \in \{0,1\}^n$; every signing of of v is in J.*

Proof: (hint) Let $A = (-\infty, 0]^{V^+ \setminus V^-} \times [0,\infty)^{V^- \setminus V^+} \times (-\infty,\infty)^{V \setminus V^+ \triangle V^-}$, where $V^+ \triangle V^- = (V^+ \setminus V^-) \cup (V^- \setminus V^+)$, and $B = 0$. By the Common Optimum Lemma $J_A \cap J_B \neq \emptyset$. Now an arbitrary $v \in J_A \cap J_B$ will do ! (It is not trivial though that v has the claimed properties: one first has to show, using Lovász's theorem, that $\text{dist}(J, A) = 0$, implying $v \in A$. Now compare A with what Lovász's theorem claims.) □

Corollary 1: *If J is a jump-system, and $h \in \mathbb{Z}^n$ is a hole in J, then there exist $x, y \in J_h : h = (x+y)/2$.* □

A jump system $J \subseteq \{0,1\}^n$ is called a *delta-matroid*. If the charcteristic vector of a subset of $\{1,\ldots,n\}$ is in J, then it is called a *feasible set*. If all feasible sets have even cardinality, the delta-matroid is said to be *even*. It is a well-known problem of Bouchet and Jackson to characterize the existence of a partition of $\{1,\ldots,n\}$ into two feasible sets of an even delta-matroid. The following corollary, conjectured by Cunningham, shows a half-integer weakening.

Corollary 2: (The feasible set double cover theorem) *Let J be a jump-system and suppose there exist two vectors in conv(J) which sum up to the all 1 vector. Then there exist $v_1, v_2, v_3, v_4 \in J$ such that $1 = (v_1 + v_2 + v_3 + v_4)/2$.*

Proof: Let $F = \{v_1 + v_2 : v_1, v_2 \in J\}$. According to a lemma of Bouchet and Cunningham [1], F is a jump system, so the previous corollary can be applied: $1 = (x+y)/2$, $(x, y \in F)$. □

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9 Monique Laurent
Positive Semidefinite Programming for Max-Cut: Geometric Results

(In Memory of Svata Poljak)

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This talk is devoted to results obtained jointly with Svata Poljak on the geometry of the semidefinite relaxation for the max-cut problem. Svata had, in fact, introduced me to this topic in September 1993 at the occasion of a one month visit in Paris and our collaboration continued thereafter.

Positive semidefinite programming has been applied successfully for several combinatorial optimization problems, in particular, for the maximum stable set problem in perfect graphs (by the work of Grötschel, Lovász and Schrijver) and, more recently, for the max-cut problem. Goemans and Williamson have shown how to design good and efficient approximation algorithms for max-cut by optimizing over the set $\mathcal{E}_n$ of $n \times n$ symmetric positive semidefinite matrices with diagonal entries one. Since then, it has been recognized that similar techniques apply for several other problems (such as max 2-sat, graph coloring, max k-cut, or maximum bisection problems) using the set $\mathcal{E}_n$ with possibly some additional linear constraints.
Our work with Svata focused, in particular, on trying to understand the geometry of the set $\mathcal{E}_n$, that we called an *elliptope* (standing for *ellip*soid and poly*tope*). Indeed, we believe that a better insight into the geometric properties of the elliptope could possibly yield refinements and improvements on the algorithmic side.

The elliptope $\mathcal{E}_n$ constitutes a linear (nonpolyhedral) relaxation of the usual cut polytope P_n. It retains several features of this polytope. For

Moreover

$$\bar{0} = (-\infty, 0]^{V^+ \setminus V^-} \times [0, \infty)^{V^- \setminus V^+} \times 0^{V^+ \cap V^-} \times (-\infty, \infty)^{V \setminus (V^+ \cup V^-)},$$

for all $v \in J_0$ and $i \in V^+ \cap V^-$, $v_i \in \{-1, 0, 1\}$, and resigning v on $V^+ \cap V^-$ in an arbitrary way we get again a vector in J_0.

Proof: (hint) If there were two maximal boxes with the stated property, then taking for every i the union and the intersection of their defining intervals, we get two boxes, A and B, $A \supseteq B$. But then $J_A \cap J_B \neq \emptyset$ by the Common Optimum Lemma, and dist(J, A) =dist$(J, 0)$ follows easily, in contradiction with the maximality of the original boxes. The other two statements also follow by applying the Common Optimum Lemma to a pair of appropriate boxes. □

Example: The Gallai-Edmonds structure theorem for matchings describes the set of maximum matchings of a graph G with the help of a partition of the vertex-set into three sets $D(G)$, $A(G)$, $C(G)$ (see [4] page 94). Let us partition $D(G)$ into two sets, $D_1(G)$, $D_2(G)$, where $D_1(G)$ is the union of components of $D(G)$ consisting of one vertex, and $D_2(G)$ is the union of the bigger components. Now let J consist of the degree vectors of subgraphs of G after subtracting the all 1 vector from each: dist$(J, 0)$ is then the deficiency of a maximum matching. (It is easy to see that we cannot get closer to the all 1 vector if besides vertices of degree 0 and 1, we also allow vertices of degree 2.) It can be checked now that $V^+ \cap V^- = D_2(G)$, $V^- \setminus V^+ = D_1(G)$, $V^+ \setminus V^- = A(G)$, $\{1, \ldots, n\} \setminus (V^+ \cup V^-) = C(G)$ (these four sets correspond to the for types of intervals in Lovász's theorem, whence they immediately define $\bar{0}$).

2. Gaps

Given $x \in \mathbb{Z}^n$ and a jump system $J \subseteq \mathbb{Z}^n$, $x \notin J$ can have two very different reasons:
1. $v \notin$conv(J).
2. $v \in$conv$(J) \setminus J$, that is, v is a *gap*.
In the first case we have a convincing certificate of the non-membership. Our goal here is to show some surprising consequences if the second case holds.

1. Lovász's structure theory

The *membership problem*: Let $v \in \mathbb{Z}^n$. Is $v \in J$ true ? It is not possible to give a 'good' (characterization) answer to this question, since that would imply a good characterization for general matroid matching. Still, Lovász's theorem provides an answer which becomes a good characterization in important special cases including matroid intersection, or graph factors.

A *box* is a set of the form $\{x \in \mathbb{R}^n : a \leq x \leq b\}$, where $a, b \in \mathbb{R}^n$, $a \leq b$. For $x, y \in \mathbb{R}^n$ we use the notation $d(x, y) = \sum_{i=1}^n |x_i - y_i|$, and for $X, Y \subseteq \mathbb{R}^n$, $d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}$ (this infimum will always be a minimum). If J is a jump system and A is a box, then let $J_A := \{x \in J : d(x, A) = d(J, A)\}$.

The following lemma is the most important simple tool for proving a certain kind of statements. It often happens that proofs are simplified by applying it with a better chain of boxes. It is actually often more convenient to use this, than the subsequent theorem.

The Common Optimum Lemma:[3] *If $A \supseteq \ldots \supseteq Z$ are boxes, then $J_A \cap \ldots \cap J_Z \neq \emptyset$.*

Proof: (hint) First, prove that an arbitrary $x \in J$ can be 'A-pushed' to J_A, that is, there exist a sequence $x_0 := x, x_1, \ldots, x_t = a \in J_A$, ($x_i \in J$), so that x_i is at one or two steps from x_{i-1} and $d(x_i, A) \leq d(x_{i-1}, A)$, $i = 1, \ldots, t$.

Second, remark that in case $A \supseteq B$, an A-push is also a B-push.

Third, finish the proof by deleting A and using induction: let $x \in J_B \cap \ldots \cap J_Z$ and apply the first proven statement to x and A; then apply the second remark to conclude. □

We state only a restricted version of Lovász's theorem, which contains though the main point, and everything we will need later. (To restrict oneself to the box $0 \in \mathbb{R}^n$ does not really restrict generality.) The folowing notation and this way of stating Lovász's theorem was introduced by Jim Geelen: let $V^+ := \{i \in \{1, \ldots, n\} :$ there exists $x \in J_0$ with $x_i > 0\}$, $V^- := \{i \in \{1, \ldots, n\} :$ there exists $x \in J_0$ with $x_i < 0\}$.

Lovász's theorem: *If $J \subseteq \mathbb{Z}^n$ is a jump system, then there exists a unique maximal box $\bar{0}$ so that $0 \in \bar{0}$ and dist$(J, \bar{0})$ =dist$(J, 0)$.*

instance, the elliptope $\mathcal{E}_n$ has vertices (that is, points on its boundary having a full-dimensional normal cone) that correspond precisely to the cuts of the complete graph K_n. Moreover, P_n and $\mathcal{E}_n$ have a lot of faces in common up to dimension $\log n$; in fact, any set of cuts in general position yields a (simplex) face for both P_n and $\mathcal{E}_n$. This geometric feature indicates that the elliptope is 'wrapped' fairly tightly around the cut polytope.

As the elliptope is a nonpolyhedral convex body, the spectrum of its face dimensions may be lacunary. This is indeed the case. All possible dimensions for the faces of $\mathcal{E}_n$ are known as well as for its polyhedral faces. In fact, $\mathcal{E}_n$ has proper faces up to dimension $\binom{n-1}{2}$ (faces of that maximum dimension being isomorphic to $\mathcal{E}_{n-1}$) and polyhedral faces up to dimension about $\sqrt{2n}$. Thus, though nonpolyhedral, $\mathcal{E}_n$ has yet large dimensional polyhedral faces.

It is possible to obtain a tighter relaxation of the cut polytope by strengthening the linear inequalities that define the elliptope. Along this line, we find the so-called gap inequalities that contain many known classes of inequalities such as triangle inequalities or hypermetric inequalities. Gap inequalities turn out, however, to be fairly complicated. Their structure is far from being well understood. It is, for instance, not known whether there exist gap inequalities yielding facets of the cut polytope with a gap greater than or equal to 2 (that is, yielding new facets not belonging to the known class of hypermetric facets).

It is well-known that every stable set problem on a graph G can be easily formulated as a max-cut problem on the graph ∇G, obtained from G by adding a new node adjacent to all nodes of G. In fact, the same connection exists between the associated polytopes and their positive semidefinite relaxations as well. Therefore, the set TH(G), which is the convex body considered by Grötschel, Lovász and Schrijver for formulating the positive semidefinite relaxation for the stable set problem on a graph G, can be seen as the intersection of the elliptope of the graph ∇G by a set of hyperplanes. In this manner we have a unified setting for the various positive semidefinite approximations considered in the literature.

Interestingly, the elliptope has also been intensively studied in connection with other problems relevant to combinatorial matrix theory.

Indeed, the so-called completion problem for positive semidefinite matrices which has received a lot of attention in the literature of linear algebra, is nothing but the problem of finding a closed-form description for projections of the elliptope $\mathcal{E}_n$. The set of entries on which one projects can be seen as the edge set of a graph G and then we have the problem of describing the elliptope $\mathcal{E}(G)$ of a graph G. Good characterizations are available for some classes of graphs; for instance, for series-parallel graphs or for graphs without certain excluded wheels as induced subgraph. The description of $\mathcal{E}(G)$ involves then certain conditions relying on the cut polytope of G.

11 András Sebő
Gaps and Jumps

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A *jump system* [1] is a set $J \subseteq \mathbb{Z}^n$ with the property that for all $x, y \in J$ and each vector x' at one step from x towards y there exists a vector $x'' \in J$ at one step from x' towards y. (We say that x' is at one step from x towards y if $x' - x$ is ± 1 times a unit vector and the non-zero coordinate of $x' - x$ has the same sign as the corresponding coordinate of $y - x$.)

Jump systems generalize, among others, matroids, delta-matroids, and degree sequences of the subgraphs of a graph (easy exercises, see also [1], [2]). Their convex hulls constitute the most general class of polyhedra for which the greedy algorithm works.

They have a property which is substantially new comparing to most objects of polyhedral combinatorics: they do not contain all the integer points of their convex hull. Those they do not contain – called *gaps* – are the subject of this talk. The study of gaps allows to cease and generalize some phenomena of matching theory, which can then also be applied elsewhere.

In the first part of the talk I explain the fundamental results of Lovász (proved at a previous meeting partially supported by DONET) [3].

These generalize the main ideas of well-known structure theorems in matching theory. Then I show some particular consequences that came out of discussions of Cunningham, Geelen and Sebő using Lovász's toolbox. (A joint article with a complete account of the results is under preparation.) Namely, it will be proved that every gap is the average of two closest points of the jump system. This result is then used to derive several consequences.

In this extended abstract I cover most of the talk, including the hints of the proofs. An interested reader might reconstruct the full proofs along these lines.

Corollary 10.0.1 *Any cutting plane proof of* $0 \geq 1$ *from the inequalities (*) – (*) has at least* $2^{\Omega((n/\log n)^{1/3})}$ *steps.*

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10 Pavel Pudlák
Lower bounds for cutting plane proofs

(abstract of a lecture)

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We prove an exponential lower bound on the length of cutting plane proofs. The proof uses an extension of a lower bound for monotone circuits to circuits which compute with real numbers and use nondecreasing functions as gates.
In cutting plane proofs we use the usual propositional variables $\bar{p}$ with the interpretation 0 = *false* and 1 = *true*. A proof line is an inequality

$$\sum_k c_k p_k \geq C,$$

where c_k and C are integers.
The **axioms** are $p_k \geq 0$ and $-p_k \geq -1$ (i.e. $0 \leq p_k \leq 1$) for every propositional variable p_k.
The rules are

1. **addition:** from $\sum_k c_k p_k \geq C$ and $\sum_k d_k p_k \geq D$ derive $\sum_k (c_k + d_k) p_k \geq C + D$;

2. **division:** from $\sum_k c_k p_k \geq C$ derive $\sum_k \frac{c_k}{d} p_k \geq \lceil \frac{C}{d} \rceil$, provided that $d > 0$ is an integer which divides each c_k;

3. **multiplication:** from $\sum_k c_k p_k \geq C$ derive $\sum_k d c_k p_k \geq dC$, where d is an arbitrary positive integer.

Cutting plane system is a refutation system: we want to refute a set of inequalities by deriving a contradiction, represented as $0 \geq 1$.

A *monotone real circuit* is a circuit which computes with real numbers and uses arbitrary nondecreasing real unary and binary functions as gates.
We say that a monotone real circuit computes a boolean function (uniquely determined by the circuit), if for all inputs of 0's and 1's the circuit outputs 0 or 1.

Theorem 1 *Let P be a cutting plane proof of the contradiction $0 \geq 1$ from inequalities*

$$\sum_k c_{i,k} p_k + \sum_l b_{i,l} q_l \geq A_i, i \in I,$$

$$\sum_k c'_{j,k} p_k + \sum_m d_{j,m} r_m \geq B_j, j \in J.$$

Suppose that all the coefficients $c_{i,k}$ are nonnegative, or all the coefficients $c'_{i,k}$ are nonpositive, then one can construct a real monotone circuit $C(\bar{p})$ such that for every 0-1 assignment α for $\bar{p}$

$C(\alpha) = 0 \quad \Rightarrow \quad \sum_k c_{i,k} a_k + \sum_l b_{i,l} q_l \geq A_i, i \in I$ *are unsatisfiable, and*

$C(\alpha) = 1 \quad \Rightarrow \quad \sum_k c'_{j,k} a_k + \sum_m d_{j,m} r_m \geq B_j, j \in J$ *are unsatisfiable.*

The size of the circuit is polynomial in the binary length of the numbers $A_i, i \in I, B_j, j \in J$ and the number of inequalities in P.
Moreover, we can construct in polynomial time a cutting plane proof of the contradiction $0 \geq 1$ from inequalities $\sum_k c_{i,k} a_k + \sum_l b_{i,l} q_l \geq A_i, i \in I$ if $C(\alpha) = 0$, respectively $\sum_k c'_{j,k} a_k + \sum_m d_{j,m} r_m \geq B_j, j \in J$ if $C(\alpha) = 1$; the length of this proof is less than or equal to the length of P.

The idea of the proof is as follows. Given an assignment for the common variables $\bar{p}$, we want to split the proof, so that we get a refutation either from q-clauses or from r-clauses. Now the rule which can mix variables $\bar{q}$ with variables $\bar{r}$ is the addition of two inequalities. Thus our strategy is not to perform this rule in such a case and keep two inequalities. In the original proof this would spoil the divisibility condition for the division rule, but as we replace variables $\bar{p}$ by an

assignment, we can treat them as a part of the constant term and we do not need the divisibility condition for them. The final inequalities in both proofs contain only constant terms and one of them is contradictory. To find out which, it suffices to compute successively the constant terms in one of the parts, say, in the $\bar{q}$ inequalities, which can be done by a monotone real circuit.
The following theorem is proved by adapting Razborov's lower bound on monotone boolean circuits [2] to monotone real circuits.

Theorem 2 *Suppose that the inputs for a monotone real circuit C are 0-1 vectors of length $\binom{n}{2}$ encoding in the natural way graphs on an n-element set. Suppose that C outputs 1 on all cliques of size m and outputs 0 on all complete $m-1$-partite graphs where $m = \lfloor \frac{1}{8}(n/\log n)^{2/3} \rfloor$. Then the size of the circuit is at least*

$$2^{\Omega((n/\log n)^{1/3})}.$$

Let n and m be given. We shall use variables $p_{i,j}$, $1 \leq i < j \leq n$ to code a graph on n vertices; variables $q_{k,i}$ will code a one-to-one mapping from an m element set into the vertices of the graph; variables $r_{i,l}$ will code an $m-1$-coloring of the graph. Hence the following inequalities express that a graph contains a clique of size m and it is $m-1$-colorable.

$$\sum_i q_{k,i} \geq 1, \quad \text{for } k = 1, \ldots, m; \qquad (*)$$

$$\sum_k q_{k,i} \leq 1, \quad \text{for } i = 1, \ldots, n; \qquad (*)$$

$$p_{i,j} - q_{k,i} - q_{k',j} \geq -1 \quad \text{for } 1 \leq i < j \leq n, 1 \leq k, k' \leq m, k \neq k'; \quad (*)$$

$$\sum_l r_{i,l} \geq 1, \quad \text{for } i = 1, \ldots, n; \qquad (*)$$

$$p_{i,j} + r_{i,l} + r_{j,l} \leq 2, \quad \text{for } 1 \leq i < j \leq n, l = 1, \ldots, m-1. \qquad (*)$$

As there are no such graphs the inequalities are inconsistent. The decision whether the first three are inconsistent or the last two are inconsistent is equivalent to the decision whether the graph coded by $p_{i,j}$'s does not contain an m-clique or it is not $m-1$-colorable. Thus combining Theorems 1 and 2 we get: