Brooks-type theorems for choosability with separation

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Abstract

We consider the following type of problems. Given a graph $G = (V, E)$ and lists $L(v)$ of allowed colors for its vertices $v \in V$ such that $|L(v)| = p$ for all $v \in V$ and $|L(u) \cap L(v)| \le c$ for all $uv \in E$, is it possible to find a "list coloring", i.e., a color $f(v) \in L(v)$ for each $v \in V$, so that $f(u) \ne f(v)$ for all $uv \in E$? We prove that every graph of maximum degree Δ admits a list coloring for every such list assignment, provided $p \ge \sqrt{5.437c\Delta}$. Apart from a multiplicative constant, the result is tight, as lists of length $\sqrt{0.5c\Delta}$ may be necessary. Moreover, for $G = K_n$ (the complete graph on n vertices) and $c = 1$ (i.e., almost disjoint lists), the smallest value of p is shown to have asymptotics $(1 + o(1))\sqrt{n}$. For planar graphs and $c = 1$, lists of length 4 suffice.

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1 Introduction

Let p, q, r be natural numbers, and $G = (V, E)$ a graph with vertex set V and edge set E. A (p, r)*-assignment* of G is a collection $\mathcal{L} = \{L(v) \mid v \in V\}$ of lists (sets) of "colors" assigned to the vertices such that

$|L(v)| = p$ for all $v \in V$, and

$|L(u) \cup L(v)| \geq p + r$ for all $uv \in E$.

An $\mathcal{L}$*-admissible q-set coloring* of G is a collection $\mathcal{F} = \{F(v) \mid v \in V\}$ such that

$F(v) \subseteq L(v)$ for all $v \in V$,

$|F(v)| = q$ for all $v \in V$, and

$|F(u) \cap F(v)| = \emptyset$ for all $uv \in E$.

The graph G is said to be *(p,q,r)-choosable* if it admits a q-set coloring for every (p, r)-assignment $\mathcal{L}$.

This concept was introduced in [3], where mainly the computational complexity issues were considered.

One can easily see by induction on the number of vertices that every graph of maximum degree Δ is $(\Delta m + m, m, 0)$-choosable for every natural number m. What is more, extending a Brooks-type theorem of Erdős, Rubin and Taylor [2], it has been proved in [5] that if each connected component in a graph of maximum degree Δ contains a block (maximal 2-connected subgraph) different from complete graphs and odd cycles, then the graph is $(\Delta m, m, 0)$-choosable. (As a matter of fact, for each vertex, a list of length m times the vertex degree suffices.)

In the present paper our goal is to show that restrictions on the intersections of lists of adjacent vertices lead to stronger sufficient conditions for the existence of list colorings. E.g., it is now well known that planar graphs are 5-choosable [4] but not always 4-choosable [6]. One of our results shows that every planar graph is $(4, 1, 3)$-choosable, i.e., if the 4-element lists assigned to adjacent vertices have at most one color in common, one can always find an admissible coloring.

We first consider complete graphs. In general, the complete graph on n vertices has choosability n. A simple observation shows that K_n is $(n - 2, 1, 1)$-choosable, i.e., requiring that adjacent vertices are assigned distinct lists guarantees the existence of a feasible coloring even if the lists are smaller than the choosability by 2. Requiring that adjacent lists have small intersections affects the choosability even more drastically. Our Theorem 2.2 shows that K_n is $(\lfloor\sqrt{n}\rfloor+2,1,\lfloor\sqrt{n}\rfloor+1)$-choosable, and we also prove that this bound is assymptotically best possible. In Section 3 we consider a variant of Brooks's theorem: we show that a graph of maximum degree Δ is $(\lceil\sqrt{5.437(\Delta-1)}\rceil,1,\lceil\sqrt{5.437(\Delta-1)}\rceil-1)$-choosable. (Up to a multiplicative constant, this result is also tight due to the bounds for complete graphs established in Section 2.) The proof of this result is nonconstructive, and we prove a constructive (but generally weaker) estimate in the last section.

In order to simplify the formulation of our results, we introduce the following notation. For a graph G and a nonnegative integer c, we denote by $\zeta(G,c)$ the minimum integer k such that G is $(k,1,k-c)$-choosable. Note that G is then $(k,1,k-c)$-choosable for every $k\geq\zeta(G,c)$ (deleting arbitrary $k-\zeta(G,c)$ colors from every list in a given k-assignment we obtain an instance of $(\zeta(G,c),1,\zeta(G,c)-c)$-list coloring, which is by definition always feasible).

2 Complete graphs

We consider complete graphs in this section. First we mention the case where the lists assigned to distinct vertices are mutually distinct.

Proposition 2.1 *For $n\geq 3$, the complete graph K_n is $(k,1,1)$-choosable if and only if $k=1$ or $k\geq n-2$.*

Proof. The case $k=1$ is trivial, a $(1,1)$-assignment is itself a proper coloring.

Suppose $k\geq n-2$ and suppose the lists $L(u),u\in V(K_n)$ are distinct. Note that for a complete graph, a list coloring is just a system of disinct representatives of the lists. We show that

$$|\bigcup_{u\in X}L(u)|\geq|X|$$

for every $X\subset V(K_n)$. This is obvious for $|X|\leq n-2$. For $|X|=n-1\geq 2$, $|\bigcup_{u\in X}L(u)|\geq k+1\geq n-1=|X|$. Finally, since an $(n-1)$-element set has only $n-1$ distinct $(n-2)$-element subsets,

$|\bigcup_{u\in X} L(u)| \geq n$ for $|X| = n$. Thus, the existence of an $\mathcal{L}$-admissible coloring follows by Hall's theorem.

For $2 \leq k \leq n-3$, we have $\binom{n-1}{k} \geq n$ and one can find n distinct k-element subsets of an $(n-1)$-element color set. Assigning these subsets to the vertices of K_n we obtain a non-feasible $(k,1)$-assignment, since the total number of colors is less than the number of vertices. $\Box$

In the sequel, we deal with the case where the lists assigned to adjacent vertices differ substantially. In particular, a $(k, k-c)$-assignment is such that the intersections of lists assigned to adjacent vertices have size at most c. Recall the definition of $\zeta(G,c)$ from the previous section. We consider $c=1$ first.

Theorem 2.2 *Let $n \geq 3$. Then $\zeta(K_n, 1) \leq \lfloor\sqrt{n - \frac{11}{4}} + \frac{3}{2}\rfloor$.*

Proof. Let $k_0 = \lfloor\sqrt{n - \frac{11}{4}} + \frac{3}{2}\rfloor$, i.e., k_0 is the smallest integer greater than $\sqrt{n - \frac{11}{4}} + \frac{1}{2}$. Suppose we are given n color sets $L_1, L_2, \ldots, L_n$ of size k_0 each, such that the intersection of any two of them has at most one element. As in the previous proof, a feasible coloring of this K_n is then a system of distinct representatives for the L_i's. We will show its existence via Hall's theorem. Let X be a subset of $\{1, 2, \ldots, n\}$, say $|X| = t$. We need to show that $|\bigcup_{i\in X} L_i| \geq t$. Suppose on the contrary that $|\bigcup_{i\in X} L_i| \leq t-1$. Consider the at most $\binom{t-1}{2}$ pairs of colors from $\bigcup_{i\in X} L_i$. Since the color sets are almost disjoint, each of these pairs belongs to at most one L_i. Hence

$$t\binom{k_0}{2} \leq \binom{t-1}{2},$$

which yields

$$tk_0(k_0-1) \leq (t-1)(t-2),$$

$$k_0(k_0-1) \leq \lfloor\frac{(t-1)(t-2)}{t}\rfloor = \lfloor t - 3 + \frac{2}{t}\rfloor \leq \lfloor n - 3 + \frac{2}{n}\rfloor = n-3.$$

It follows that

$$k_0^2 - k_0 - (n-3) \leq 0$$

and hence

$$k_0 \leq \frac{1+\sqrt{4n-11}}{2} < k_0$$

a contradiction. $\square$

The following theorem shows that the bound given in Theorem 2.2 is the best possible for infinitely many values of n:

Theorem 2.3 *For any prime power q, $\zeta(K_{q^2+1}, 1) = q + 1$.*

Proof. Take an affine plane of order q. It has $q+1$ classes of parallel lines, q lines per class, q points per line. Any two nonparallel lines share exactly one point. Forget arbitrary $q-1$ of the lines and view the remaining q^2+1 lines as color sets (the points of the plane are viewed as colors). In this assignment any two lists share at most one color, and the assignment does not allow an admissible coloring, since the total number q^2 of colors is less than the number (q^2+1) of sets. $\square$

Theorem 2.2 also gives the correct asymptotics for $\zeta(K_n, 1)$:

Theorem 2.4 *We have* $lim_{n\to\infty}\frac{\zeta(K_n,1)}{\sqrt{n}} = 1.$

Proof. The proof is again based on finite planes. Given an n (which is supposed to be large enough), let q be the smallest prime such that $q^2 \geq n$. It is well known that for any real ε and any natural number n sufficiently large with respect to ε, there is a prime between n and $(1+\varepsilon)n$. Therefore $q^2 - n = o(n)$ and $q = \sqrt{n} + o(\sqrt{n})$. Take an affine plane of order q, choose a class of parallel lines and remove $\lceil\frac{q^2-n+1}{q}\rceil = o(q)$ of them together with all points of these lines. Consider the remaining q^2 (shortened) lines as color sets. We have $q^2 \geq n$ color sets, any two of them share at most one color, and they have equal size $s = q - \lceil\frac{q^2-n+1}{q}\rceil = q - o(q)$. Their union has $q^2 - q\lceil\frac{q^2-n+1}{q}\rceil \leq q^2 - q^2 + n - 1 = n - 1$ colors, and therefore it is impossible to choose distinct colors from the sets. This shows that $\zeta(K_n, 1) > s = q - o(q) = \sqrt{n} - o(\sqrt{n})$, and so the statement follows from Theorem 2.2. $\square$

For the case of $c > 1$, a lower bound on $\zeta(K_n, c)$ is given by the following result.

Theorem 2.5 *For every integer $c \geq 2$,*

$$\liminf_{n\to\infty} \frac{\zeta(K_n,c)}{\sqrt{n}} \geq \sqrt{\lfloor \frac{c}{2} \rfloor}.$$

Proof. Suppose n is large enough (compared to c) and let q be the largest prime such that $q^2\lfloor\frac{c}{2}\rfloor < n$. Again, $\sqrt{\frac{n}{\lfloor\frac{c}{2}\rfloor}} - q = o(\sqrt{n})$ and $q > \sqrt{\frac{n}{2\lfloor\frac{c}{2}\rfloor}}$. For n large enough, $q > c$. We consider the point set $M = \{0, 1, \ldots, q\lfloor\frac{c}{2}\rfloor - 1\} \times GF(q)$. For any quadratic function f on $GF(q)$, let $M_f = \{(x, f(x \bmod q)) : 0 \leq x \leq q\lfloor\frac{c}{2}\rfloor - 1\}$, i.e., M_f consists of the graph of the function f and its translates. Let $\mathcal{F}$ be a set of n distinct quadratic functions over $GF(q)$. Since the total number of quadratic functions is $q^3 > c\frac{n}{2\lfloor\frac{c}{2}\rfloor} \geq n$, we can find such $\mathcal{F}$. We set $M(\mathcal{F}) = \{M_f : f \in \mathcal{F}\}$.

Given a complete graph with n vertices, we associate its vertices with the functions in $\mathcal{F}$ and view the sets $M_f, f \in \mathcal{F}$ as lists assigned to the vertices. The size of each list is $q\lfloor\frac{c}{2}\rfloor$. Any two distinct quadratic functions on $GF(q)$ coincide in at most two values, and hence $|M_f \cap M_g| \leq 2\lfloor\frac{c}{2}\rfloor \leq c$. Thus, our lists form a $(q\lfloor\frac{c}{2}\rfloor, q\lfloor\frac{c}{2}\rfloor - c)$-assignment. Since the total number $q^2\lfloor\frac{c}{2}\rfloor$ of colors is less than n, this assignment is not feasible. Therefore $\zeta(K_n, c) > q\lfloor\frac{c}{2}\rfloor = \sqrt{n\lfloor\frac{c}{2}\rfloor} - o(\sqrt{n})$. $\square$

It will follow from Theorem 3.1 that this bound (as a function of n and c) is tight up to a multiplicative constant.

3 Choosability versus maximum degree

In this section we prove a strengthening of Brooks's theorem, under the condition that the intersections of the lists assigned to adjacent vertices are bounded.

Theorem 3.1 *Let $\Delta(G)$ be the maximum degree of a vertex in a graph G. Then $\zeta(G,c) \leq \lceil\sqrt{2\mathrm{e}c(\Delta(G)-1)}\rceil$, where* $\mathrm{e} = 2.718\ldots$ *is the Euler constant.*

Proof. Suppose lists of size $k \geq \sqrt{2\mathrm{e}c(\Delta(G)-1)}$ are assigned to the vertices of G in such a way that $|L(u) \cap L(v)| \leq c$ whenever

$uv \in E(G)$. Consider a random coloring of G, that is every vertex u is colored with any color from $L(u)$ independently with probability $\frac{1}{k}$. For every edge $e \in E(G)$ we define the event $B(e) =$ $\langle$*the endpoints of* e *are colored with the same color*$\rangle$. We have

$$Prob(B(e)) \leq \frac{c}{k^2}$$

for every edge $e \in E(G)$. The line graph of G, $L(G) = (E(G), \{ef : e \cap f \neq \emptyset\})$ serves as the dependency graph – for every edge e, the event $B(e)$ is totally independent of the events $B(f)$ for edges f which are nonadjacent to e in the line graph $L(G)$. The maximum degree in the line graph is at most $2\Delta(G) - 2$. Due to the choice of k,

$$(2\Delta(G) - 2) \cdot \frac{c}{k^2} \leq \frac{1}{\mathrm{e}}$$

and therefore by the Lovász Local Lemma ([1]),

$$Prob(G \text{ is properly colored}) = Prob(\bigwedge_{e \in E(G)} \overline{B(e)}) > 0.$$

It follows that there exists an admissible coloring. □

As seen from Theorem 2.4 for complete graphs, this bound is best possible up to a multiplicative constant. Namely, we have the following corollary.

Corollary 3.2 *For every integer $c \geq 2$,*

$$\sqrt{\lfloor \frac{c}{2} \rfloor} \leq \liminf_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \leq \limsup_{n\to\infty} \frac{\zeta(K_n, c)}{\sqrt{n}} \leq \sqrt{2\mathrm{e}c}.$$

Proof. The first inequality is just repeated from Theorem 2.5; the last one follows from Theorem 3.1 since $\Delta(K_n) = n - 1$. □

Problem 1. Does there exist a limit $L = \lim_{n,c\to\infty} \frac{\zeta(K_n,c)}{\sqrt{nc}}$? If yes, determine the limit. (Our bounds give $\sqrt{\frac{1}{2}} \leq L \leq \sqrt{2\mathrm{e}}$.)

4 Constructive bounds

The sufficient condition for choosability in the previous section was proved by probabilistic methods, and so far it is not known whether there exists a polynomial-time deterministic coloring algorithm when the conditions hold. Here we present another upper bound on the necessary length of lists, in terms of maximum degree Δ. This bound gives an improvement for not too large Δ and also admits a polynomial-time coloring algorithm.

Theorem 4.1 *Let $m = \max_{H\subseteq G}\lceil\frac{|E(H)|}{|V(H)|}\rceil$ for a graph G. Then $\zeta(G,c) \leq cm+1$, i.e., G is $(cm+1, 1, c(m-1)+1)$-choosable and for any given assignment, an admissible coloring can be found in polynomial time.*

Proof. Suppose that m satisfies the assumptions and that lists $L(u), u \in V(G)$ of size $cm+1$ are given. It is well known that then the edges of G can be oriented so that every vertex has outdegree at most m. Such an orientation can be found in polynomial time, e.g. by bipartite matching techniques. Let A be the set of arcs in such an orientation. For every vertex u, delete those colors from $L(u)$ that occur in the lists of the successors of u, i.e., define $L'(u) = L(u)\setminus\bigcup_{v:uv\in A} L(v)$. Assuming that lists assigned to adjacent vertices have intersections of size at most c, we have $|\bigcup_{v:uv\in A} L(v)| \leq mc$ and hence $L'(u)$ is nonempty for every $u \in V(G)$. Coloring u with an arbitrary color from $L'(u)$ yields a feasible coloring. □

Remark. In the same way, under the assumptions of Theorem 4.1, one can prove that G is $(cm+t, t, c(m-1)+t)$-choosable for any $t \geq 1$.

Corollary 4.2 *Let $\Delta = \Delta(G)$ and $k \geq \frac{\Delta+2}{2}$. Then G is $(k, 1, k-1)$-choosable and for any given assignment, a feasible coloring can be found in polynomial time.*

Proof. For any graph G and any of its subgraphs H, $\frac{|E(H)|}{|V(H)|} \leq \frac{\Delta(H)}{2} \leq \frac{\Delta(G)}{2}$. Hence, $m \leq k-1$ and the result follows from Theorem 4.1 for $c = 1$. □

Corollary 4.3 *Every planar graph is* $(4, 1, 3)$*-choosable, and a feasible coloring can be found in polynomial time.*

Proof. Planar graphs satisfy $|E(H)| \leq 3|V(H)| - 6$, hence $m = 3$. The result follows from Theorem 4.1 for $c = 1$. $\square$

Examples of planar non-4-choosable graphs are known such that the infeasible list assignments assign distinct lists to adjacent vertices. This leaves a challenging open problem:

Problem 2. Is every planar graph $(4, 1, 2)$-choosable?

Remark. For $\Delta \leq 16$, Corollary 4.2 gives a better bound for $\zeta(G, 1)$ than Theorem 3.1.

References

[1] P. Erdős, L. Lovász: Problems and results on 3-chromatic hypergraphs and related questions, in: Infinite and Finite Sets (A. Hajnal, L. Lovász and V. Sós, eds.), Colloq. Math. Soc. J. Bolyai 10, North-Holland, Amsterdam, 1974, pp. 609–627

[2] P. Erdős, A. Rubin, H. Taylor: Choosability in graphs, Congr. Numer. 26 (1979), 125–157

[3] J. Kratochvíl, Zs. Tuza, M. Voigt: Choosing subsets from color sets (to appear)

[4] C. Thomassen: Every planar graph is 5-choosable, J. Combin. Theory Ser. B 62 (1994), 180–181

[5] Zs. Tuza, M. Voigt: On a conjecture of Erdős, Rubin and Taylor, Tatra Mountains Math. Publ. 9 (1996), 69–82

[6] M. Voigt: List colourings of planar graphs, Discrete Math. 120 (1993), 215–219