Graph Designs, Hadamard Matrices and Geometric Configurations *

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Abstract

Motivated by graphs arising from special geometrical configurations (equiangular line systems and balanced incomplete block designs) we investigate graph equations of the type $(2n-1)G(2n, n-1) = (n-1)K_{2n}$. This rather special equation proves to be general enough

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to cover Hadamard matrices and a new concept: *matchability* of Hadamard matrices. We show that several known constructions and matrix operations produce matchable Hadamard matrices, thus supporting our general conjecture that all Hadamard matrices are matchable.

1 Introduction

The 1-skeleton of the icosahedron and the great icosahedron are isomorphic graphs. A search for explanation of this well known fact has led us to the beautiful theory of equiangular line systems and related graphs developed by J.J.Seidel [7]. This theory centers around the notion of graph switching. Motivated by this we consider graph equations of the following types:

$$K_{2n} = 2G(2n, n-1) + nK_2 \tag{1}$$
$$(n-1)K_{2n} = (2n-1)G(2n, n-1) \tag{2}$$

(here and throughout the paper, $G(n, m)$ denotes an m-regular graph on n vertices). We believe that this particular graph equation (which can be also interpreted as graph designs in the sense of Hell and Rosa [3]) is interesting even in rather special cases of the simplest $G(2n, n-1)$ graphs. For $G(2n, n-1) = 2K_n$, we show that (2) is equivalent to the existence of Hadamard matrices of order $2n$. For $G(2n, n-1) = K_{n,n} - nK_2$ (sometimes called the Hiraguchi graph), the situation is more involved. In this case we relate the problem of solvability of (2) to a new concept of *matchability* of Hadamard matrices. While this notion will be defined later on in a more technical way, here is a concise definition of the matchability concept:

Let $H = (a_{ij})_{i,j=1}^{4m}$ be a Hadamard matrix of order $4m$ in standard form (i.e., the first row and the first column consist of $+1$'s). Delete the first column of H to get a $4m \times (4m-1)$ matrix A, in which every column contains exactly $2m$ $+1$'s and $2m$ -1's, and in which every two rows have Hamming distance $2m$. We say that the matrix A (and the Hadamard matrix H) are *matchable* if for every column j there exists an involution (i.e., a matching) $\phi_j : \{1, 2, \ldots, 2m\} \to \{1, 2, \ldots, 2m\}$ such that

$$a_{ij} a_{\phi_j(i),j} = -1$$

and such that for every pair $i < i'$ of distinct rows there is a unique j such that $\phi_j(i) = i'$.

Presently we do not know if there exist non-matchable Hadamard matrices. In fact, we show that several well known standard constructions of Hadamard matrices yield matchable matrices. Particularly, we show that the tensor product of matchable Hadamard matrices is matchable (Theorem 9) and that some Hadamard matrices derived from quadratic residues are matchable (Theorem 7). We believe that problems raised in this context are of independent interest.

Our work builds upon the work of several authors and in the first part is necessarily partly of expository nature. One should mention again Seidel's papers [7] which provide the original motivation. A nice exposition of Seidel's theory is the recent paper [9]. Another related area is presented by graph designs, which were first defined and studied by Hell and Rosa [3].

2 Geometric examples and Seidel's switching

Let $L = \{L_1, L_2, \ldots, L_n\}$ be a system of $n > d$ equiangular lines in the d-dimensional euclidean space passing through the origin. (A system of lines is equiangular if all angles among the lines are the same and smaller than $\frac{\pi}{2}$.) Let $X = \{x_1, x_2, \ldots, x_{2n}\}$ be the intersection of these lines with the unit sphere centred in the origin, i.e., X contains 2 points of unit distance from the origin on each of the lines. We define three graphs on the vertex set X:

$G_L = (X, \{x_i x_j | \angle x_i 0 x_j < \frac{\pi}{2}\})$,
$\widetilde{G_L} = (X, \{x_i x_j | \angle x_i 0 x_j > \frac{\pi}{2}\})$,
$M_L = (X, \{x_i x_j | x_i = -x_j\})$.

The graph M_L is a perfect matching and we denote it by nK_2. The edge set of the complete graph on $2n$ vertices is the disjoint union of the edge sets of G_L, $\widetilde{G_L}$ and nK_2, we say that K_{2n} decomposes into G_L, $\widetilde{G_L}$ and nK_2. Formally, this fact is expressed in the graph equation

$$K_{2n} = G_L + \widetilde{G_L} + nK_2. \tag{3}$$

Note that both G_L and $\widetilde{G_L}$ are $(n-1)$-regular graphs on $2n$ vertices, i.e., they are both $G(2n, n-1)$ graphs in our notation.

Example 1. Let L_1, L_2, L_3 be the longest diagonals of a regular hexagon $x_1x_2x_3x_4x_5x_6$ in E^2, centered in the origin (cf. Figure 1). These lines form an equiangular line system, as any two of them determine an angle of $\frac{\pi}{3}$. The edges of the hexagon form our graph $G_L = C_6$ and the shorter diagonals determine $\widetilde{G_L} = 2K_3$. Equation (3) thus becomes

$$K_6 = C_6 + 2K_3 + 3K_2.$$

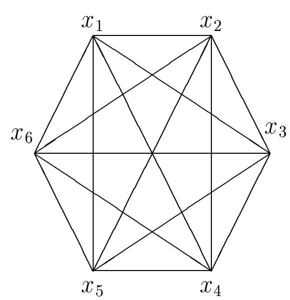

Figure 1:

Example 2. Consider a regular icosahedron in the 3-dimensional space. The lines passing through the longest diagonals form a system of 6 equiangular lines. The edges of the icosahedron form our G_L. In this case, the graph $\widetilde{G_L}$ formed by the shorter diagonals, the great icosahedron, turns out to be isomorphic to the icosahedron itself. Hence equation (3) becomes

$$K_{12} = 2Ico + 6K_2$$

(here Ico denotes the graph of the icosahedron).

Comparing Examples 1 and 2, we see that the icosahedron provides a solution to equation (1), while the hexagon does not (C_6 is not isomorphic to $2K_3$). (This does not mean that *any* system of 6 equiangular lines would provide a solution to (1) – one can construct systems of equiangular lines

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Example. Consider $H_2 = \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}$ (this is the only Hadamard matrix whose order is not divisible by 4). Clearly, H_2 is negative matchable (but not positive matchable). Since H_4 is both positive and negative matchable, it follows from Theorem 9 that $H_8 = H_2 * H_4$ is negative matchable. The matrix H_8 is positive matchable as well, as it follows from an extension of Lemma 4.4:

Lemma 4.7 *If a Hadamard matrix H' is both positive and negative matchable then $H_2 * H'$ is positive matchable.*

Proof As in the proof of Theorem 9, let M' be positive and N' negative matching for H'. Let us first define $\mathcal{M}_{(k,l)}$ for $l > 1$ as in the previous proof:

$$\mathcal{M}_{(1,l)} = \{\{(i,j),(i,j')\} : \{j,j'\} \in M'_l\},$$
$$\mathcal{M}_{(2,l)} = \{\{(1,j),(2,j')\} : \{j,j'\} \in M'_l\}.$$

So far every pair $\{(i,j),(i',j')\}, i,i' = 1,2,\ j,j' = 1,2,\ldots,m$ occurs in exactly one $\mathcal{M}_{(k,l)}$, except for the pairs $\{(1,j),(2,j)\}, j = 1,2,\ldots,m$. To cover also these pairs, we modify $\mathcal{M}$ in column $(1,2)$ and then complete the matching in column $(2,1)$. Thus we set

$$\overline{\mathcal{M}}_{(1,2)} = \{\{(1,j),(2,j)\} : j = 1,2,\ldots,m\},$$
$$\overline{\mathcal{M}}_{(2,1)} = \mathcal{M}_{(1,2)},$$
$$\overline{\mathcal{M}}_{(k,l)} = \mathcal{M}_{(k,l)},\ (k,l) \neq (1,2),(2,1).$$

It is straightforward to verify that $\overline{\mathcal{M}}$ is a positive matching for $H_2 * H'$. □

Some of our other results on matchability of Hadamard matrices can be extended to positive matchability as well. E.g., it is true that for every $n = 2^k$, the Hadamard matrix $H_n = H_2^k$ is positive matchable. One may actually ask if every negative matchable Hadamard matrix is also positive matchable.

Another daring conjecture would be that every column balanced $2n \times (2n-1)$ matrix with row distances $\geq (n-1)$ is matchable. The truth is, however, that we were not able to find any counterexample to this conjecture.

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in higher-dimensional space such that G_L and $\widetilde{G_L}$ are non-isomorphic.) At this point it seems that the theory of equiangular lines may be a source of many solutions to (1). Our first goal is to study the pattern suggested by the icosahedron and great icosahedron, which leads to the following natural question: *When are the graphs G_L and $\widetilde{G_L}$ isomorphic?*

This problem fits into Seidel's theory and can be answered within this framework. Recall that given any graph $G = (V,E)$ and a subset $A \subset V$, the graph $S(G,A) = (V, \{xy|xy \in E, |A \cap \{x,y\}| \neq 1\} \cup \{xy|xy \notin E, |A \cap \{x,y\}| = 1\})$ is obtained from G by *switching* the vertices of A. Graphs G, H are called *switching equivalent* if H is isomorphic to $S(G,A)$ for some $A \subset V(G)$.

Given a system L of n equiangular lines, choose an n-element subset Y of X so that Y contains one point of each line L_i. Denote H_L the subgraph of G_L induced by Y, i.e., $H_L = (Y, \{xy|\angle x0y < \frac{\pi}{2}\})$. (The graph H_L is not unique, it depends on the choice of Y. However, it follows from Seidel's theory that all graphs H_L are switching equivalent and form a switching class. We use this notation since we are only interested in properties invariant under switching.) It is worth mentioning that Seidel proved [7] that every graph H can be obtained in this way as H_L for some equiangular line system L. Now we can formulate the answer to our question:

Theorem 1 *The graphs G_L and $\widetilde{G_L}$ are isomorphic if and only if H_L is switching equivalent to its complement.*

Thanks to Theorem 1 we can reverse our strategy: In order to find all solutions of (1), we

1. search for all graphs H on n vertices which are switchable to their complements;
2. represent H as H_L for an equiangular line system L;
3. build G_L which then becomes a solution to (1).

Let us note that G_L can be constructed from H in a purely combinatorial way (without the intermediate step of equiangular line systems). Given a graph H we define G_H as follows:

$$V(G_H) = \{(h,i)|h \in V(H), i = 1,2\},$$

$$E(G_H) = \{(h,i)(h',i)|hh' \in E(H)\} \cup \{(h,i)(h',j)|i \neq j, h \neq h', hh' \notin E(H)\}.$$

It is clear that for any n-vertex graph H, G_H is a $G(2n, n-1)$ graph and $G_H = G_L$ if $H = H_L$. Seidel's result says that any G_H can be also constructed geometrically via equiangular line systems. The following simple observation is important for the proof of Theorem 1 ($\bar{H}$ denotes the complement of H):

Observation 2.1 *For any graph H,*

$$G_{\bar{H}} = \widetilde{G_H}.$$

□

Theorem 1 then follows directly from this Observation and from the following Theorem (in the proof, as well as in the rest of the paper, n is reserved to denote the number of vertices of H):

Theorem 2 *For any two graphs H and H', the graphs G_H and $G_{H'}$ are isomorphic if and only if H and H' are switching equivalent.*

Proof The 'if' part is easy. If $f : V(H) \to V(H')$ is an isomorphism of $S(H, A)$ onto H' for some $A \subset V(H)$, then $g : V(G_H) \to V(G_{H'})$ defined by

$$g(h,i) = \begin{cases} (f(h), i) & \text{if } h \in A \\ (f(h), 3-i) & \text{if } h \in V(H) - A \end{cases}$$

is an isomorphism of G_H onto $G_{H'}$.

For the 'only if' part, suppose that G_H and $G_{H'}$ are isomorphic. It then follows from the first part of the theorem that G_{H^*} and $G_{H'^*}$ are isomorphic for any H^* switching equivalent to H, and H'^* switching equivalent to H'.

For a graph G and two of its vertices u, v, we write $u \sim_G v$ if u and v have the same closed neighborhoods. This $\sim_G$ is an equivalence relation and each equivalence class induces a complete subgraph of G. Let $H_0 = S(H, A)$ be a graph switching equivalent to H such that the equivalence $\sim_{H_0}$ is the coarsest possible. We claim that H_0 has the following property:

$$\forall x, y : \; xy \notin E(H_0) \;\Rightarrow\; \exists z : \{xz, yz\} \subset E(H_0) \;\vee\; \{xz, yz\} \cap E(H_0) = \emptyset \quad (4)$$

Indeed, if not then H_0 contains vertices x, y such that every other vertex is adjacent either to x or to y. If $[x]_{\sim_{H_0}}$ and $[y]_{\sim_{H_0}}$ are the equivalence

Theorem 9 *If Hadamard matrices H and H' are both positive matchable then their tensor product $H * H'$ is positive matchable as well. If one of the matrices H and H' is both positive and negative matchable and the other one is negative matchable then their tensor product $H * H'$ is negative matchable as well.*

Proof Let $H = (a_{ik})_{i,k=1}^n$ and $H' = (a'_{jl})_{j,l=1}^m$ be Hadamard matrices in standard form (namely, $a_{i1} = a'_{j1} = +1$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$). Recall that the tensor product $H * H'$ is the $mn \times mn$ matrix with entries

$$b_{(i,j),(k,l)} = a_{ik} \cdot a'_{jl}.$$

It is well known and easy to see that the tensor product of Hadamard matrices is again a Hadamard matrix.

For each i, k, denote by $H'_{i,k}$ the $m \times m$ submatrix of $H * H'$ formed by all entries $b_{(i,j),(k,l)}, j, l = 1, \ldots, m$. Clearly, $H'_{i,k}$ is a copy of H' if $a_{i,k} = +1$, and $H'_{i,k} = -H'$ if $a_{i,k} = -1$. Let $M_k, N_k, k = 2, 3, \ldots, n$ ($M'_l, M'_l, l = 2, 3, \ldots, m$) be matchings of H (H', resp.) that satisfy (ii'). We define column-wise matchings $\mathcal{M}_{(k,l)}$ of $H * H'$ as follows:

$$\mathcal{M}_{(1,l)} = \{\{(i,j),(i,j')\} : \{j,j'\} \in M'_l, i = 1, 2, \ldots, n\}, l > 1,$$
$$\mathcal{M}_{(k,1)} = \{\{(i,j),(i',j)\} : \{i,i'\} \in M_k, j = 1, 2, \ldots, m\}, k > 1,$$
$$\mathcal{M}_{(k,l)} = \{\{(i,j),(i',j')\} : \{i,i'\} \in N_k, \{j,j'\} \in N'_l, \}, k > 1, l > 1.$$

It is easy to see that $\mathcal{M}$ satisfies (ii'): for any pair of distinct rows of $H * H'$, say (i,j) and (i',j'), there exists exactly one column (k,l) such that $\{(i,j),(i',j')\} \in \mathcal{M}_{(k,l)}$ – if $i = i'$, this is $(k,l) = (1,l)$ for the unique l such that $\{j,j'\} \in M'_l$; if $j = j'$, this is $(k,l) = (k,1)$ for the unique k such that $\{i,i'\} \in M_k$; and for $i \neq i', j \neq j'$, this is (k,l) for the unique k, l such that $\{i,i'\} \in N_k$ and $\{j,j'\} \in N'_l$.

Consider any $\{(i,j),(i',j')\} \in \mathcal{M}_{(k,l)}$. We have

$$b_{(i,j),(k,l)} \cdot b_{(i',j'),(k,l)} = a_{ik} \cdot a'_{jl} \cdot a_{i'k} \cdot a'_{j'l} = a_{ik} \cdot a_{i'k} \cdot a'_{jl} \cdot a'_{j'l}$$

and we see that $\mathcal{M}$ is positive if all M, N, M', N' are positive (and also if M, M' are positive and N, N' negative) and $\mathcal{M}$ is negative if M, M' are negative and exactly one of N, N' is positive and the other one negative. □

Proof The Hadamard matrix of order two H_2 is matchable and therefore by induction H_n (defined recursively as $H_{2m} = H_2 * H_m$) is both matchable for $n = 2^k$. Then apply Observation 4.1. □

Corollary 4.6 *For every $n = (q+1)\cdot 2^k$, such that $k \geq 0$ and $q \equiv 3 \bmod 4$, $G = K_{n,n} - nK_2$ solves (2).*

Proof Similar to the previous proof, starting with a matchable Hadamard matrix of order $q+1$ (Theorem 7) and using Lemma 4.4. □

Another well known fact says that the tensor product of Hadamard matrices is again a Hadamard matrix. A similar result is true about matchability of Hadamard matrices. We need to introduce two auxiliary notions. Let A be a column balanced ± 1 matrix of size $2n \times (2n-1)$. We say that A is *positive matchable* (resp. *negative matchable*) if there exist column-wise matchings M_k, $k = 1, 2, \ldots, 2n-1$ such that

(i') $\{i,j\} \in M_k$ implies $A_{i,k} \cdot A_{j,k} = +1$ (resp. $A_{i,k} \cdot A_{j,k} = -1$) and

(ii') for any two distinct i, j, there is exactly one k such that $\{i,j\} \in M_k$.

We say that a Hadamard matrix in the standard form is positive matchable (negative matchable) if its truncation is positive (negative) matchable. Note that a Hadamard matrix is matchable if and only if it is negative matchable in the sense of this definition.

Example. The Hadamard matrix H_4 is both positive and negative matchable, as shown in Fig. 6.

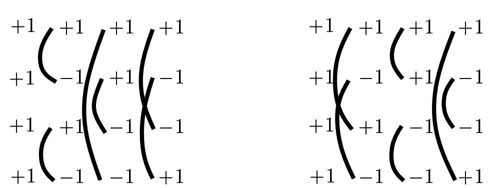

Figure 6: Positive and negative matchability of H_4.

classes containing x and y, respectively, then in $S(H_0, [y]_{\sim_{H_0}})$ their union $[x]_{\sim_{H_0}} \cup [y]_{\sim_{H_0}}$ is a larger equivalence class, while the other equivalence classes remain unchanged. This would contradict the choice of H_0.

Next we show the key observation. If H_0 fulfils (4) then the distance of (x,i) and (y,j) in G_{H_0} is ≥ 3 if and only if $i \neq j$ and $x \sim_{H_0} y$. To see this, observe first that for $i \neq j$, $d_{G_{H_0}}((x,i),(y,j)) \leq 2$ if and only if $xy \notin E(H_0)$ or there is a z which is adjacent to exactly one of x, y. None of these may occur for $x \sim_{H_0} y$. For $i = j$ and $xy \in E(H_0)$, we have $d_{G_{H_0}}((x,i),(y,j)) = 1$, and for $i = j$ and $xy \notin E(H_0)$, the existence of z guaranteed by (4) implies $d_{G_{H_0}}((x,i),(y,j)) = 2$. Finally, let $i \neq j$ and $x \not\sim_{H_0} y$. Then $d_{G_{H_0}}((x,i),(y,j)) = 1$, provided $xy \notin E(H_0)$. If $xy \in E(H_0)$ then the nonequivalence of x and y implies the existence of a vertex z which is adjacent to exactly one of x, y, say to x. Then $\{(x,i)(z,i),(z,i)(y,j)\} \subset E(G_{H_0})$ and $d_{G_{H_0}}((x,i),(y,j)) = 2$.

Now we complete the proof of the theorem. We let $H_0' = S(H', B)$ be a graph switching equivalent to H' with a coarsest possible equivalence $\sim_{H_0'}$. Since $G_H \cong G_{H'}$ by the assumption and $G_H \cong G_{H_0}$ and $G_{H'} \cong G_{H_0'}$ according to the first part of this theorem, we have $G_{H_0} \cong G_{H_0'}$.

Let $f : V(G_{H_0}) \to V(G_{H_0'})$, $f(x,i) = (f_1(x,i), f_2(x,i))$ be an isomorphism of G_{H_0} and $G_{H_0'}$. Since f preserves distances, H_0' must have equivalence classes of $\sim_{H_0'}$ of the same sizes as H_0, and $f_2(x,i) = f_2(y,i)$ whenever $x \sim_{H_0} y$. It follows that for $C = \{x | f_2(x,1) = 2\}$, the mapping g defined by

$$g(x) = f_1(x,1),\ x \in V(H_0)$$

is an isomorphism of H_0 onto $S(H_0', C)$. Hence H_0 and H_0', and consequently also H and H', are switching equivalent.

□

Examples.

(i) There is no graph with 3 vertices switchable to its complement, and thus (1) has no solution for $n = 3$. More generally, it is easy to show that if H is switchable to its complement then n is even or $n \equiv 1 \bmod 4$. Therefore (1) has no solution for $n \equiv 3 \bmod 4$.

(ii) If G_1 and G_2 are selfcomplementary graphs then their disjoint union $G_1 + G_2$ is switchable to its complement ($G_1 \oplus G_2$ - the complete join or Zykov sum of G_1 and G_2). Thus $C_5 + K_1$ is switchable to its complement W_5

(the 5-wheel) and this pair gives rise to the solution

$$K_{12} = 2Ico + 6K_2$$

of (1).

One can ask how difficult it is to test the condition of Theorem 1. It is well known that switching equivalence is isomorphism-complete [1], and we have the similar result about switchability to complements:

Theorem 3 *The following problem*
Instance: A graph H.
Question: Is H switchable to its complement?
is isomorphism-complete.

Proof Being a special case of switching equivalence, it is obvious that our problem is not harder then isomorphism test. To show that it is at least that hard, we use the well known fact that testing selfcomplementarity of graphs is isomorphism-complete:

Given a graph G, in order to decide whether $G \cong \bar{G}$, we consider the complete join of two copies of G, $G' = G \oplus G = (V(G) \times \{0,1\}, \{(h,i)(h',j) | i \neq j \text{ or } hh' \in E(G)\})$. Without loss of generality we assume that both G and its complement $\bar{G}$ are connected (otherwise they could not be isomorphic). We claim that $G \cong \bar{G}$ if and only if G' is switching equivalent to $\bar{G}'$.

If G is selfcomplemetary then $\bar{G}' = \bar{G} + \bar{G} \cong G + G = S(G', V(G) \times \{0\})$.

Suppose $\bar{G}' \cong S(G', A)$ for some $A \subset V(G')$. Since $\bar{G}'$ consists of two connected components of size $|V(G)|$ and $S(G', A)$ has all edges between $A \cap (V(G) \times \{0\})$ and $A \cap (V(G) \times \{1\})$ (and between $(V(G) \times \{1\}) - A$ and $(V(G) \times \{0\}) - A)$, there must be no edges between $(V(G) \times \{0\}) - A$ and $(V(G) \times \{0\}) \cap A$ (and between $(V(G) \times \{1\}) - A$ and $(V(G) \times \{1\}) \cap A)$ in $S(G', A)$. Then for $i = 0, 1$, $A \subset (V(G) \times \{i\})$ or $A \cap (V(G) \times \{i\}) = \emptyset$ (otherwise $\bar{G}$ would not be connected), and the only possibilities are $A = V(G) \times \{0\}$ or $A = V(G) \times \{1\}$. In both cases $S(G', A) = G + G$ and $S(G', A)$ is isomorphic to $\bar{G}' = \bar{G} + \bar{G}$ only if $G \cong G'$. □

Our approach often exhibits an interesting additional property of the solution of (1). The following is the key observation:

Proposition 2.2 *If n, the number of vertices of H, is even then the chromatic index of G_H is $n-1$ (i.e., G_H is Vizing class 1).*

Let $\nu \in GF(q)$ be a quadratic non-residue such that $\nu - 1$ is a quadratic residue in $GF(q)$. It is easy to see that if q is odd then one can always find such an element. In column α we match the first row A^* with row A^α. In the matrix R each column is a permutation of all the elments of $GF(q)$. In each column we match the row containing a quadratic residue α with the row containing the element $\nu\alpha$. Clearly in A the first row contains a 1 while the second row contains a -1, hence this is indeed a perfect matching in each column. It is easy to see that for each $\alpha \in GF(q)$ the rows A^* and A^α are matched once. If $\alpha - \beta$ is a quadratic residue we need to show that there are two columns γ and δ in which these two rows are matched. First let $\gamma = \frac{\nu\beta - \alpha}{\nu - 1}$. A simple calculation shows that

$$\gamma - \alpha = \frac{\nu(\beta-\alpha)}{\nu-1} \quad (6)$$
$$\alpha - \gamma = \nu(\beta - \gamma) \quad (7)$$

Since $\alpha - \beta$ and $\nu - 1$ are quadratic residues, from (6) we get that $\gamma - \alpha$ is a quadratic non-residue and from (7) we see that $\beta - \gamma$ is a quadratic residue. It then follows from (7) that A^α and A^β are matched in column γ. Similarly, if we let $\delta = \frac{\nu\alpha - \beta}{\nu - 1}$ we get that A^α and A^β are also matched in column δ and $\delta \neq \gamma$. In these calculations we use the fact that if $q \equiv 1 \bmod 4$ then if α is a quadratic residue then so is $-\alpha$. □

Lemma 4.4 *If a Hadamard matrix H is matchable then $H_2 * H$ is also matchable.*

Proof Let M be a negative matching for H. We define $\mathcal{M}_{(k,l)}$ in the following way:

$$\mathcal{M}_{(1,l)} = \{\{(1,j),(2,j')\} : \{j,j'\} \in M_l\},\ l > 1,$$
$$\mathcal{M}_{(2,l)} = \{\{(i,j),(i,j')\} : \{j,j'\} \in M_l, i = 1,2\},\ l > 1,$$
$$\mathcal{M}_{(2,1)} = \{\{(1,j),(2,j)\} : j = 1,2,\ldots,m\}.$$

It is easy to see that $\mathcal{M}$ is a matching for $H_2 * H$. □

Corollary 4.5 *For every $n = 2^k$, $k \geq 1$, $G = K_{n,n} - nK_2$ solves (2).*

Proof We say that row H^α is matched with row H^β in column γ if $\{\alpha,\beta\} \in M_\gamma$ (where M_γ, $\gamma \in GF(q)$ are the columnwise matchings that witnes matchability of H). As noted before, in each column of the truncated $(q+1)\times q$ matrix we need to construct a perfect matching such that every pair of distinct rows is matched in some column. We set

$$M_\beta = \{\{*,\beta\}\} \cup \{\{\alpha, 2\beta-\alpha\} : \alpha \in GF(q) - \{\beta\}\}, \beta \in GF(q).$$

In other words, in column β, we match H^* with row H^β and we match row H^α, $\alpha \neq \beta$, with row $H^{2\beta-\alpha}$. It is simple to describe this matching in terms of the matrix R. Each column of R is a permutation of the q elements of $GF(q)$. In each column we match the row containing α with the row containing $-\alpha$. The row containing 0 is matched with the added top row. Since $q \equiv\ 3 \bmod 4$, $\chi(-1) = -1$ and hence $\chi(\alpha-\beta) = -\chi(\beta-\alpha)$ and since $H_{\beta,\beta} = -1$ our match indeed matches a $+1$ with a -1. Also if $\alpha \neq \alpha'$, $2\beta-\alpha \neq 2\beta-\alpha'$. Therefore in each column we get a perfect matching.

We show that every pair of rows are matched in some column: Row H^* is matched with row H^β in column β and rows H^α and H^γ are matched in column $\beta = \frac{\alpha+\gamma}{2}$. □

When $q \equiv\ 1 \bmod 4$ the Hadamard matrix H derived from Q is more complex. We first form the matrix:

$$S_{q+1} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1}^T & Q \end{pmatrix}.$$

This is a symmetric conference matrix of order $q+1$. Let A be the $(q+1)\times q$ matrix obtained from $S_{q+1} - I_{q+1}$ by deleting the first column. We know that A is a column balanced matrix of type $(q+1)\times q$ and the Hamming distance between the first row and any other row is $\frac{q+1}{2}$. It is easy to see that $dist_H(A^\alpha, A^\beta) = \frac{q+1}{2} + \chi(\alpha-\beta)$.

Theorem 8 *The matrix A is matchable.*

Proof We denote the first row of A by A^*. All other rows and all columns of A will be indexed by the corresponding elements of $GF(q)$. We need to show that in each column we can select a perfect matching so that in all these q matchings the pair $\{A^*, A^\alpha\}$ is matched once for each α and the pairs $\{A^\alpha, A^\beta\}$ are matched twice whenever $\chi(\alpha-\beta) = 1$.

Proof Let $M_1, M_2, \ldots, M_{n-1}$ be a 1-factorization of the complete graph on the vertex $V(H)$. Define $F_t = \{(h,i)(h',j) | (h,i)(h',j) \in E(G_H), hh' \in M_t\}$ for $t = 1,2,\ldots,n-1$. Then $F_1, F_2, \ldots, F_{n-1}$ is a 1-factorization of G_H. □

Incidently, this result proves that the chromatic index of Icosahedron is 5. Combining 5 solutions of (1)

$$K_{12} = 2Ico + 6K_2$$

in such a way that the five matchings $6K_2$ factorize a copy of Icosahedron, we obtain the following solution of (2):

$$5K_{12} = 11Ico.$$

This is a particular instance of graph designs introduced and studied by Hell and others [3]. (A graph design $kG = tH$ is a system $\mathcal{H}$ of k subgraphs of a graph H, each of them isomorphic to G, such that every edge of H belongs to exactly t graphs of $\mathcal{H}$. In this sense Balanced Incomplete Block Designs, which obviously motivated the definition of Graph Designs, are graph designs of type $bK_n = \lambda K_v$. It is natural to talk about Spanning Graph Designs if in addition to $kG = tH$, the order of G is the same as the order of H. Thus the solutions of (2) are Spanning Graph Designs of a special type.)

Theorem 1 and Proposition 2.2 immediately yield:

Theorem 4 *Let H be a graph of even order switchable to its complement. Then G_H provides a solution to (2), i.e.*

$$(2n-1)G_H = (n-1)K_{2n}.$$

□

Remarks Theorem 4 suggests the following problems, some of which may be of an independent interest:

Problem 1. Suppose that H is switching equivalent to its complement. Does it follow that $\chi'(G_H) = n-1$?

By Proposition 2.2, this is a problem for odd n's only. Thus the smallest example is C_5, for which the problem has an affirmative solution depicted in Figure 2. Note that C_5 and $K_1 + P_3$ are switching equivalent and hence the

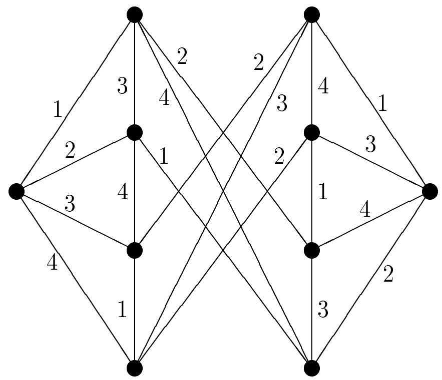

Figure 2: Edge-coloring of $G_{K_1+P_3}$

graphs $G_{K_1+P_3}$ and G_{C_5} are isomorphic. Thus G_{C_5} presents a solution to (2) for $n = 5$.

Let us also remark that Problem 1 has a negative solution if the switchability condition is dropped. For instance, for $H = K_n$ (and more generally for every nearly complete graph H with more than $\binom{n}{2} - \frac{n-1}{2}$ edges) we have $\chi'(G_H) = n$. This motivates the following:

Problem 2. Suppose that every graph switching equivalent to H has at most $\binom{n}{2} - \frac{n-1}{2}$ edges. Is it then true that $\chi'(G_H) = n - 1$?

The condition on the number of edges is obviously necessary: For any H' switching equivalent to H, the graphs G_H and $G_{H'}$ are isomorphic. Thus H' as a subgraph of $G_{H'}$ has itself chromatic index $n-1$ and $|E(H')| \leq \binom{n}{2} - \frac{n-1}{2}$ follows. An evidence towards the validity of our conjecture may be the result of Plantholt [6] which says that every graph on n vertices with exactly $\binom{n}{2} - \frac{n-1}{2}$ edges is $(n-1)$-edge-colorable (an $(n-1)$-edge-coloring of G_H then follows).

Let us list two more problems directly related to Theorem 4:

Problem 3. Suppose that H is switchable to its complement. Does G_H

we show that Hadamard and Conference matrices derived from quadratic residues over $GF(q)$ (Paley's construction, for more details see [2]) can be used to construct matchable column balanced matrices of type $2n \times (2n-1)$. We first attend to Hadamard matrices. Since the truncation of a Hadamard matrix (in standard form) is uniquely defined, we say that a Hadamard matrix is *matchable* if its truncation is matchable. We conjecture that every Hadamard matrix is matchable:

Problem 9. Is every Hadamard matrix matchable? Or at least, is it true that if a Hadamard matrix of order $4m$ exists then there also exists a matchable Hadamard matrix of the same order?

We support our conjecture by showing that the well known standard constructions of Hadamard matrices give matchable matrices.

Let q be a prime power. We denote by $\mathbf{1} = (1, 1, \ldots, 1)$ the all-one vector (of appropriate length, which will be clear from context), and by I_n the identity matrix of size $n \times n$. Let $\chi(x)$ denote the quadratic residue character over $GF(q)$ (i.e., $\chi(0) = 0$, $\chi(x) = +1$ iff $x = y^2$ for some $y \in GF(q) - \{0\}$ and $\chi(x) = -1$ otherwise). We define the following matrices over $GF(q)$:

$$R = (\alpha - \beta)_{\alpha,\beta \in GF(q)},$$

$$Q = (\chi(\alpha - \beta))_{\alpha,\beta \in GF(q)},$$

$$H'_{q+1} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1}^T & Q \end{pmatrix} + I_{q+1}.$$

As shown in [2], H'_{q+1} is a Hadamard matrix of order $q+1$ if $q \equiv 3 \bmod 4$. The equivalent Hadamard matrix in standard form is

$$H_{q+1} = \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{1}^T & -Q - I_q \end{pmatrix}.$$

We denote the first row of this matrix by H^* and the other rows by $H^\alpha, \alpha \in GF(q)$.

Theorem 7 *If $q \equiv 3$ mod 4 then the Hadamard matrix $H = H_{q+1}$ is matchable.*

Similarly, if $|A_i| = |A_j| = n-1$, then $|A_i \cap A_j| = \frac{1}{2}(|A_i|+|A_j|-|A_i \div A_j|) = \frac{1}{2}(2n-2-2\lceil\frac{n-1}{2}\rceil) = \lfloor\frac{n-1}{2}\rfloor$.

If $|A_i| = n$ and $|A_j| = n-1$, then $|A_i \cap A_j| = \frac{1}{2}(|A_i|+|A_j|-|A_i \div A_j|) = \frac{1}{2}(2n-1-2(\lfloor\frac{n-1}{2}\rfloor+1)) = \lceil\frac{n-1}{2}\rceil$.

Hence the affirmative solution to the next problem would imply the existence of column balanced $2n \times (2n-1)$ matrices with large row distances:

Problem 8. Do there exist $n-1$ sets $A_i \subset \{1,2,\ldots,2n-1\}$, for any $n \geq 2$, such that

(i) $|A_i| \in \{n-1, n\}$ for $i = 1,2,\ldots,n-1$,

(ii) for $i \neq j$, $|A_i \cap A_j| = \begin{cases} \lceil\frac{n}{2}\rceil & \text{if } |A_i| = |A_j| = n, \\ \lfloor\frac{n-1}{2}\rfloor & \text{if } |A_i| = |A_j| = n-1, \\ \lceil\frac{n-1}{2}\rceil & \text{if } |A_i| \neq |A_j|? \end{cases}$

A slightly different construction is based on conference matrices. Recall that a conference matrix C of order n is an $n \times n$ orthogonal matrix with diagonal entries 0 and all other entries ± 1. We say that a conference matrix C is in *standard form* if all entries in the first row and first column are $+1$ (except the first entry which is 0). It is easy to see that if C is a conference matrix in standard form then except for the first row and column each row and each column consists of 1 zero, $\frac{n-2}{2}$ $-1's$ and $\frac{n}{2}$ $+1's$.

Observation 4.3 *Let C be an $n \times n$ conference matrix in standard form, $B = C - I_n$ and A be the truncation of B. Then A is a column balanced matrix of type $n \times (n-1)$ with row distances $\geq \frac{n}{2} - 1$.*

Proof Every column of C (except the first one) has $\frac{n-2}{2}$ $-1's$ and $\frac{n}{2}$ $+1's$, and so every column of A has $\frac{n}{2}$ $-1's$ and $\frac{n}{2}$ $+1's$. Similarly, every row of A (except the first one) has $\frac{n}{2}$ $-1's$ and hence the Hamming distance of the first and any other row is $\frac{n}{2}$. Any other pair of rows of C differ in $\frac{n+2}{2}$ positions, and hence the rows of A (after replacing one zero in each by -1) will differ in at least $\frac{n-2}{2}$ positions. □

4.2 Matchability of Hadamard and Conference matrices

In this section we show that certain classical combinatorial structures yield matchable column balanced matrices of type $2n \times (2n-1)$. In particular

solve (2)?

This is true for all even n, and by inspection of cases also for $n = 5, 9, 13, \ldots$.

Problem 4. Suppose G is a solution of $(2n-1)G(2n, n-1) = (n-1)K_{2n}$. Does $\chi'(G) = n-1$?

3 Hadamard matrices

In this section we start analyzing particular solutions of the equation (2) which are not induced by switching equivalence via Theorem 4. First we consider the case $H = K_n$ and hence $G_H = K_n + K_n$. We have the following slightly surprising connection to the existence of Hadamard matrices:

Theorem 5 *Equation (2) has a solution $G = K_n + K_n$ if and only if there exists a Hadamard matrix of order $2n$.*

Proof Assume that $G_1 + G_2 + \ldots + G_{2n-1} = (n-1)K_{2n}$, $G_i \cong K_n + K_n$, is a solution of (2). Let $V(K_{2n}) = V(G_i) = \{1, 2, \ldots, 2n\}$. For each $j = 1, 2, \ldots, 2n-1$, define an indicator function $f_j : \{1, 2, \ldots, 2n-1\} \to \{+1, -1\}$ by

$$f_j(x) \begin{cases} +1 & \text{if } x \text{ is in the same component of } G_j \text{ as vertex 1} \\ -1 & \text{otherwise.} \end{cases}$$

Now form an $2n \times (2n-1)$ matrix $A = (a_{ij})$ with $a_{ij} = f_j(i)$.

Consider two rows r_m, r_k of A. The edge mk of K_{2n} appears in exactly $n-1$ of the graphs G_j which means that the inner product of rows r_m and r_k is -1. Thus if we extend A to a square matrix by adding an all 1 column we get a Hadamard matrix.

As one can clearly reverse the argument, we get the statement of the theorem. □

Remarks. Note that Hadamard matrices of order $2n$ exist only if n is even and thus the solutions of type $G = K_n + K_n$ do not provide a counterexample to Problem 4.

Characterization of orders of Hadamard matrices is currently considered a tough and important problem of contemporary combinatorics. Therefore the problem of determining all solutions $G = G(2n, n-1)$ of equation (2) seems to be very hard and currently beyond the reach of standard combinatorial methods.

As mentioned earlier, equation (2) is a special case of Graph Designs. Given a graph G, let us define k_G as the minimum t for which the equation

$$aG \;=\; tK_{|V(G)|} \tag{5}$$

has a solution for some a. (For this definition, G does not have to be regular.) For $G = G(2n, n-1)$, the degrees of G and K_{2n} are relatively prime and hence $k_G \geq n-1$. Equation (2) is thus asking for graphs with minimum possible t. Note also that k_G is finite for every G – obviously $t_G \leq 2|E(G)| \cdot (|V(G)|-2)!$ (taking all $|V(G)|!$ copies of G on its vertex set). The following problem is then of particular interest:

Problem 5. Determine upper bounds on $k_{K_n+K_n}$. In particular, is $\frac{k_{K_n+K_n}}{n-1}$ bounded by an absolute constant?

Another more general extremal problem suggests itself:

Problem 6. Give estimates on $k_n = \max\{k_G : |V(G)| = n\}$. Is it an exponential function of n?

4 Bipartite case and Hadamard matchability

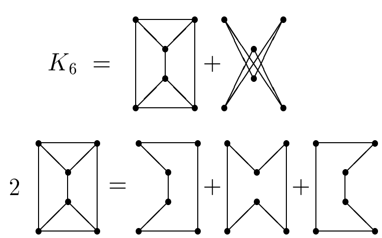

Figure 3: The prism

$C_{2m+1+i} = -B_i$ are $4m+1$ and the distances of the remaining pairs of rows from B and $-B$ are $d_{i,2m+1+j} = 4m+1-d_{i,j} = 2m+1$. Hence C is a column balanced matrix with row distances $\geq 2m$. □

Example Matrix A from Figure 4 is a truncated Hadamard matrix H_4. The same Hadamard matrix yields the following 6×5 matrix via the second construction:

$$\begin{pmatrix} +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & -1 & -1 & +1 \\ +1 & -1 & +1 & -1 & +1 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & +1 & +1 & -1 \\ -1 & +1 & -1 & +1 & -1 \end{pmatrix}.$$

The second construction actually suggests a more general construction which might prove the existence of column balanced $2n \times (2n-1)$ matrices with row distances $\geq (n-1)$ regardless of the existence of Hadamard matrices:

Observation 4.2 *If A is a ± 1 matrix of size $n \times (2n-1)$ with row distances $n-1$ or n, then the tensor product B of A and $\binom{+1}{-1}$ is a $2n \times (2n-1)$ column balanced matrix with row distances $\geq n-1$.*

Proof Obviously, B is column balanced. The row distances within A and within $-A$ are $\geq (n-1)$. The row distances of the complementary rows in A and $-A$ are $2n-1$ and the distances of the remaining pairs from A and $-A$ are $\geq 2n-1-n = n-1$. □

If there exists a matrix A as in the observation, we may suppose that the first row contains all symbols -1. It follows that every other row contains $n-1$ or n symbols $+1$. Then we may view the rows of A as characteristic vectors of sets $A_i \subset \{1, 2, \ldots, 2n-1\}$. Thus $A_1 = \emptyset$ and $|A_i| \in \{n-1, n\}$ for $i = 2, 3, \ldots, n$. The Hamming distance conditions then imply unique restrictions on the cardinalities of the intersections of A_i's:

If $|A_i| = |A_j| = n$, then $|A_i \cap A_j| = \frac{1}{2}(|A_i| + |A_j| - |A_i \div A_j|) = \frac{1}{2}(|A_i| + |A_j| - d_H(A_i, A_j)) = \frac{1}{2}(2n - 2\lfloor \frac{n}{2} \rfloor) = \lceil \frac{n}{2} \rceil$, as the symmetric difference of two sets of the same cardinality is even, and hence $d_H(A_i, A_j) = 2\lfloor \frac{n}{2} \rfloor$ in this case.

1. Construct a $2n \times (2n-1)$ column balanced matrix A such that the Hamming distance of any two rows is $\geq n-1$;

2. Show that A is matchable and use the matchings to derive a solution.

It is surprising that even Step 1 is not obvious for every n. We give two methods for construction of column balanced matrices with large row distances in Subsection 4.1 and we then pay closer attention to matchability of matrices constructed in this way in the last subsection. Observing the difficulties in constructing column balanced matrices with large row distances, we feel that the following generalization may be of some interest on its own:

Problem 7. Given m, n and integers e_{ij}, $i, j = 1, 2, \ldots, 2n$ such that $0 \leq e_{ij} \leq m$ for every $i \neq j$ and $\sum_{i \neq j} e_{ij} = mn^2$. Does there exist a column balanced matrix A of type $2n \times m$ such that the Hamming distance of any two rows r_i, r_j is e_{ij}? State necessary and sufficient conditions for the existence of such a matrix A.

4.1 Column balanced matrices

We show two general constructions of column balanced matrices which are based on well known combinatorial structures.

Theorem 6 *If there exists a Hadamard matrix of order $4m$ then column balanced matrices of size $2n \times (2n-1)$ with row distances $\geq (n-1)$ exist for both $n = 2m$ and $n = 2m+1$.*

Proof Suppose H_{4m} is a Hadamard matrix of order $4m$ in the standard form, i.e., the first row and column contain only symbols $+1$. Let A be the truncation of H_{4m}, i.e., the matrix obtained from H_{4m} by deletion of the all+1 column. Then A is column balanced and every two rows have inner product -1, hence any two rows coincide in $2m-1$ coordinates and differ in $2m$ coordinates. Thus A is a $4m \times (4m-1)$ column balanced matrix with row distances $2m \geq (2m-1)$.

Now take any $2m+1$ rows of H_{4m} and construct a matrix B by adding an all+1 column to these rows. Let C be the tensor product of B and $(+1, -1)$ (C is the $(4m+2) \times (4m+1)$ matrix whose first $2m+1$ rows form B and last $2m+1$ rows form $-B$). The row distances within the first or last $2m+1$ rows are $2m$, the distances of the complementary rows $C_i = B_i$ and

In this section we consider the next simplest case of $G(2n, n-1) = K_{n,n} - nK_2$ (the complete bipartite graph minus a perfect matching, sometimes called the Hiraguchi graph) which arises as G_H for the edgless graph H on n vertices.

Example. Consider again $n = 3$. We have $K_{3,3} - 3K_2 = C_6$. Since the complement of C_6 is the prism which can be double covered by 3 copies of C_6 (cf. Figure 3), we have

$$5C_6 = 2K_6.$$

Thus we see that not every solution of equation (2) is induced by equiangular lines via solutions of (1) and Theorem 4.

In order to analyze the feasible values of n for which $G = K_{n,n} - nK_2$ solves (2), we introduce two notions about matrices. First, a ± 1 matrix is called *column balanced* if every column contains the same number of symbols $+1$ and -1. Let A be a column balanced $2n \times m$ matrix and let $d_{i,j}$ denote the Hamming distance of the i-th and j-th row of A. Counting the sum of these distances columnwise, we see that

$$\sum_{i,j \in \{1,2,\ldots,2n\}} d_{i,j} = m \cdot n^2.$$

We say that A is *matchable* if it is possible to equalize the row distances by removing a perfect matching in each column, i.e., if there exist m sets $M_k \subset \binom{\{1,2,\ldots,2n\}}{2}$, each containing n disjoint pairs, such that

(i) $\{i, j\} \in M_k$ implies $A_{i,k} \cdot A_{j,k} = -1$ and

(ii) $d_{i,j} - |\{k : \{i,j\} \in M_k\}| = \dfrac{m(n-1)}{2n-1}$.

(The matchings are understood as a device to reduce distances of rows of A. If the i-th and j-th row differ in the k-th coordinate, then the occurrence of $\{i, j\}$ in M_k reduces the distance d_{ij} by one. The total reduction is then mn and the sum of the reduced distances is $mn^2 - mn$. Hence $\frac{m(n-1)}{2n-1} = \frac{mn^2 - mn}{\binom{2n}{2}}$ is the average reduced row distance in A. Obviously, the number of columns of a matchable matrix is divisible by $2n-1$ and a necessary condition for a $2n \times m$ matrix to be matchable is that the Hamming distance of any two rows is $\geq \frac{m(n-1)}{2n-1}$.) The role of these notions is captured by the following observation:

Observation 4.1 *The graph $K_{n,n} - nK_2$ solves equation (2) if and only if there exists a column balanced matchable matrix of type $2n \times (2n-1)$.*

Proof Let $G_1, G_2, \ldots, G_{2n-1}$ be subgraphs of K_{2n} isomorphic to $K_{n,n} - nK_2$ such that every edge of K_{2n} belongs to exactly $n-1$ of them. Assume $V(K_{2n}) = \{1, 2, \ldots, 2n\}$ and define a $2n \times (2n-1)$ matrix A by

$$A_{ik} = \begin{cases} +1 & \text{if } i \text{ lies in the same part of} \\ & G_k \text{ as } 1 \\ -1 & \text{otherwise.} \end{cases}$$

Then A is obviously column balanced. The matchings M_k such that $G'_k = G_k \cup \{ij | \{i,j\} \in M_k\} \simeq K_{n,n}$ satisfy (i-ii) and A is matchable, since $\frac{m(n-1)}{2n-1} = n-1$ for $m = 2n-1$.

On the other hand, the graphs G_k can be read out straightforwardly from columns of a column balanced matchable $2n \times (2n-1)$ matrix. □

Example. The 4×3 matrix

$$A = \begin{pmatrix} +1 & +1 & +1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix}$$

is column balanced and all row distances are 2. The matchings $M_1 = \{\{1,3\},\{2,4\}\}, M_2 = \{\{1,4\},\{2,3\}\}, M_3 = \{\{1,2\},\{3,4\}\}$ depicted in Figure 4 show that A is matchable. Therefore $K_{2,2} - 2K_2 = C_4$ solves (2) for $n = 2$.

This A is not the only matchable matrix of size 4×3. Another example is

$$B = \begin{pmatrix} +1 & +1 & +1 \\ +1 & -1 & +1 \\ -1 & +1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

with matchings $M_1 = \{\{1,4\},\{2,3\}\}, M_2 = \{\{1,4\},\{2,3\}\}, M_3 = \{\{1,3\},\{2,4\}\}$ (cf. Fig. 4 right). Note that this example is cosiderably different – the row distances are not all the same (we have $d_{12} = d_{34} = 0, d_{13} = d_{24} = 2$ and $d_{14} = d_{23} = 3$).

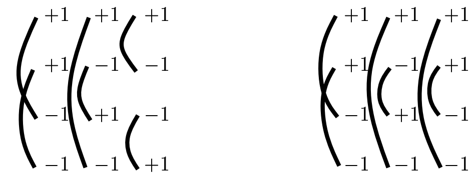

Figure 4: Column balanced matchable matrices of type 4×3.

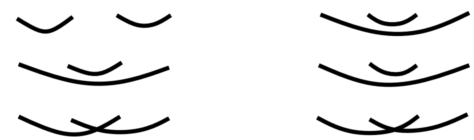

Figure 5: Fisches Nachtgesang.

Incidentally, in the transposed form the pattern of Figure 4 goes back to Ch. Morgenstern (cf. [5]):

Another example is the incidence matrix of the Fano plane (incidences are +1) completed by an all+1 row to the following 8×7 matrix:

$$A = \begin{pmatrix} +1 & +1 & +1 & +1 & +1 & +1 & +1 \\ +1 & +1 & +1 & -1 & -1 & -1 & -1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 \\ +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ -1 & -1 & +1 & -1 & +1 & +1 & -1 \end{pmatrix}$$

This matrix is column balanced and the row distances all equal 4. It's matchability is left to the reader.

Observation 4.1 suggests how to construct solutions of (2) for $G = K_{n,n} - nK_2$ in two steps: