Graphs maximal with respect to hom-properties

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Abstract

For a simple graph H, $\rightarrow H$ denotes the class of all graphs that admit homomorphisms to H (such classes of graphs are called *hom-properties*). We investigate hom-properties from the point of view of the lattice of hereditary properties. In particular, we are interested in characterization of maximal graphs belonging to $\rightarrow H$. We also provide a description of graphs maximal with respect to reducible hom-properties and determine the maximum number of edges of graphs belonging to $\rightarrow H$.

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1 Definitions and notations

All graphs considered in this paper are finite and simple (without multiple edges or loops), and we use standard notation [3]. In particular, K_n denotes the complete graph on n vertices, C_n is the cycle of length n, $G \cup H$ denotes the disjoint union of graphs G and H, $\omega(G)$ is the maximum clique size of G and $\chi(G)$ is the chromatic number of G.

The *join* $G + H$ of two graphs G and H is the graph consisting of the disjoint union of G and H and all the edges between $V(G)$ and $V(H)$. If a graph G is a join we say that it is *decomposable.* A graph that is not decomposable is called *indecomposable.* It is easy to see that a graph G is decomposable if and only if its complement $\overline{G}$ is not connected. Then G is the join of the complements of the components of $\overline{G}$. Thus every decomposable graph G can be expressed in a unique way as the join of indecomposable graphs.

We denote by $\mathcal{I}$ the class of all finite simple graphs. A *graph property* is a non-empty isomorphism-closed subclass of $\mathcal{I}$. (We also say that a graph has the property $\mathcal{P}$ if $G \in \mathcal{P}$.) A property $\mathcal{P}$ of graphs is called *hereditary* if it is closed under subgraphs, i.e., if $H \subseteq G$ and $G \in \mathcal{P}$ imply $H \in \mathcal{P}$. A property $\mathcal{P}$ is called *additive* if it is closed under the disjoint union of graphs, i.e., if every graph has property $\mathcal{P}$ provided all of its connected components have this property.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be any properties of graphs. A *vertex* $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$*-partition* of a graph G is a partition $(V_1, V_2, \ldots, V_n)$ of $V(G)$ such that for each $i = 1, 2, \ldots, n$, the induced subgraph $G[V_i]$ has the property $\mathcal{P}_i$. The composition $\mathcal{P}_1 \circ \mathcal{P}_2 \circ \ldots \circ \mathcal{P}_n$ is defined as the class of all graphs having a vertex $(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n)$-partition (for more details see [1], [8]).

A *homomorphism* of a graph G to a graph H is a mapping f of the vertex set $V(G)$ into $V(H)$ which preserves the edges, i.e., such that $e = \{u, v\} \in E(G)$ implies $f(e) = \{f(u), f(v)\} \in E(H)$. By the symbol $f(G)$ we shall denote the graph with the vertex set

$$f(V(G)) = \{f(v) \in V(H) | v \in V(G)\}$$

and the edge set

$$f(E(G)) = \{f(e) \in E(H) | e \in E(G)\}.$$

If a homomorphism of G to H exists, we say that G is *homomorphic* to H and write $G \to H$. One can easily see that $\chi(G) \leq \chi(H)$ in such a case.

A *core of a graph* G is any subgraph $G^{'}$ of G for which $G \to G^{'}$ while G fails to be homomorphic to any proper subgraph of $G^{'}$. It can be easily seen that up to isomorphism every finite graph has a unique core which will be denoted by $C(G)$ (see e.g. [4]). A graph G is a *core*, if G is a core for itself, i.e. $G \cong C(G)$.

A *hom-property* is any class $\to H = \{G \in \mathcal{I} | G \to H\}$. We say that a graph G *generates* the hom-property $\to H$ whenever $\to H = \to G$.

For any graph $G \in \mathcal{I}$ with vertex set $V(G) = \{v_1, v_2, \ldots v_n\}$, we define a *multiplication* $G^{::}$ of G in the following way:

1. $V(G^{::}) = W_1 \cup W_2 \cup \ldots W_n$,
2. for each $1 \leq i \leq n$: $|W_i| \geq 1$,
3. for any pair $1 \leq i < j \leq n$: $W_i \cap W_j = \emptyset$,
4. for any $1 \leq i \leq j \leq n$, $u \in W_i, v \in W_j$: $\{u, v\} \in E(G^{::})$ if and only if $\{v_i, v_j\} \in E(G)$.

The sets $W_1, W_2, \ldots, W_n$ are called the *multivertices* corresponding to vertices $v_1, v_2, \ldots, v_n$, respectively. The condition 4 immediately yields that $W_1, W_2, \ldots W_n$ are independent sets and any two vertices belonging to the same multivertex have identical neighborhoods. Furthermore, it is not difficult to see that $G^{::}$ is homomorphic to G. In order to emphasize the structure of $G^{::}$ we also use the notation

$G^{::}(W_1, W_2, \ldots, W_n)$. If there is no danger of confusion, we denote by $G^{::}(k)$ the multiplication $G^{::}(W_1, W_2, \ldots, W_n)$ of G with $|W_i| = k$ for $i = 1, 2, \ldots, n$.

2 Hom-properties

Observation 2.1 *Let φ be a surjective homomorphism of G to H, where $|V(G)| = |V(H)|$. Then $|E(G)| \leq |E(H)|$.*

Proof As φ is a homomorphism, it preserves the edges. The condition $\varphi(V(G)) = V(H)$ implies that no two edges of G are identified in $\varphi(G)$. Hence $|E(G)| \leq |E(H)|$. □

Lemma 2.2 *Let G, H be graphs and let φ be a homomorphism of G to H. Then*

$$\varphi(G + H) = \varphi(G) + \varphi(H).$$

Proof Let u be any vertex of G and v any vertex of H. By the definition, the edge $\{u, v\}$ is in $E(G + H)$. As φ is a homomorphism, the edge $\{\varphi(u), \varphi(v)\}$ is in $\varphi(G + K)$. It immediately follows that the vertex sets $\varphi(V(G))$ and $\varphi(V(H))$ are disjoint and there are all possible edges between $\varphi(V(G))$ and $\varphi(V(H))$. Thus $\varphi(G + H) = \varphi(G) + \varphi(H)$. □

The next useful characterization of cores was proved in [7].

Proposition 2.3 *The graph $G + H$ is a core if and only if G and H are cores.*

Hom-properties can be given in various ways, for example the property $\to C_6$ is the same as the property $\to C_{38}$. A standard way is to describe a hom-property by a core:

Proposition 2.4 *For any graph H, its core $C(H)$ generates $\to H$.*

Proof By the definition, there exists a homomorphism $\psi : H \longrightarrow C(H)$. If $G \in \to H$, then there exists a homomorphism $\varphi : G \longrightarrow H$. Then the composition of φ and ψ is a homomorphism of G to $C(H)$. Conversely, if $G \in \to C(H)$ then there is a homomorphism $\varphi : G \longrightarrow C(H)$. The composition of φ and the identity mapping is a homomorphism of G to H. □

According to the previous proposition, we can assume in the sequel that any hom-property $\to H$ is given by a core H.

Proposition 2.5 *For any graph $H \in \mathcal{I}$, the hom-property $\to H$ is hereditary and additive.*

Proof If $G \in \to H$, then there exists a homomorphism $f : G \longrightarrow H$. For any $G^* \subseteq G$, the mapping f restricted to $V(G^*)$ is a homomorphism of G^* to H. Therefore, $G^* \in \to H$ and $\to H$ is hereditary.

For $G_1, G_2 \in \to H$, let f_1, f_2 be homomorphisms of G_1 and G_2 to H, respectively. Then the mapping $f : G_1 \cup G_2 \longrightarrow H$ defined by

$$f(v) = \begin{cases} f_1(v) & \text{for } v \in V(G_1), \\ f_2(v) & \text{for } v \in V(G_2) \end{cases}$$

is a homomorphism of $G_1 \cup G_2$ to H. Hence, $\to H$ is also additive. □

For any property $\mathcal{P} \neq \mathcal{I}$, the number $c(\mathcal{P}) = \max\{k | K_{k+1} \in \mathcal{P}\}$ is finite. It is called the *completeness* of $\mathcal{P}$ (see [1]).

Proposition 2.6 *For any graph $H \in \mathcal{I}$, $c(\to H) = \omega(H) - 1$.*

Proof The homomorphic image of the complete graph K_n is again a complete graph of the same order. Thus, $c(\to H) = \omega(H) - 1$. □

The following assertions recapitulate some facts related to the position of selected hom-properties in the lattice of additive hereditary properties of graphs (see [1]). We use the notation $\mathcal{O} = \{G \in \mathcal{I} | G \text{ is edgeless}\}$, i.e., $\mathcal{O} = \to K_1$.

Proposition 2.7 *For graphs $H, H^* \in \mathcal{I}$, we have:*

1. $\to H \subseteq \to H^*$ *if and only if* $H \to H^*$,
2. $\to K_n = \mathcal{O}^n$,
3. $\to H \subseteq \mathcal{O}^{\chi(H)}$,
4. *if H contains at least one edge then* $\mathcal{O}^2 \subseteq \to H$.

According to the terminology used in extremal graph theory, we say that a property $\mathcal{P}$ is *degenerate* if it has a bipartite forbidden graph. The following proposition states that there is just one degenerate hom-property.

Proposition 2.8 *A property $\to H$ is degenerate if and only if $\to H = \mathcal{O}$.*

Proof If $\to H \neq \to K_1$ then the graph H contains at least one edge and, by Proposition 2.7, all bipartite graphs belong to $\to H$. Thus, no bipartite graph is forbidden for $\to H$, and $\to H$ is not degenerate.

On the other hand, it is easy to see that $\to K_1$ consists of all graphs with chromatic number one, and therefore it is degenerate. □

It turns out that multiplications of graphs play an important role in characterization of graphs maximal with respect to hom-properties. Hence we present some of their fundamental properties.

Lemma 2.9 *Any multiplication $H^{::}$ of an indecomposable graph H is indecomposable as well.*

Proof Let us denote by $v_1, v_2, \ldots, v_n$ the vertices of H and let $W_1, W_2, \ldots, W_n$ be the corresponding multivertices of $H^{::}$. Suppose, to the contrary, that $H^{::}$ is decomposable, i.e., there exist graphs H_1 and H_2 such that $H^{::} = H_1 + H_2$. As any multivertex W_j is an independent set in $H^{::}$, it is either a subset of $V(H_1)$ or a subset of

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Proof As the complete graph $K_{\omega(H)}$ is a subgraph of H, any graph G with the chromatic number at most $\omega(H)$ is homomorphic to H. It implies $\chi(\to H) \geq \omega(H) + 1$.

On the other hand, $K_{\omega(H)+1}$ is not homomorphic to H but each proper subgraph of $K_{\omega(H)+1}$ has chromatic number at most $\omega(H)$ and hence belongs to $\to H$. Therefore, $K_{\omega(H)+1} \in \mathbf{F}(\to H)$ and $\chi(\to H) \leq \omega(H) + 1$. □

It is worth to mentioned that the previous assertion corresponds to the general result obtained for reducible hereditary properties in [9].

Corollary 5.3 *Let* $\to H$ *be any hom-property. Then*

$$ex(n, \to H) = \left(1 - \frac{1}{\omega(H)}\right)\binom{n}{2} + o(n^2).$$

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$V(H_2)$. Let J_1 (J_2) be the sets of indices j such that $W_j \subseteq V(H_1)$ ($W_j \subseteq V(H_2)$, respectively). It is easy to see that for any choice of vertices $u_1, u_2, \ldots, u_n$ from $W_1, W_2, \ldots, W_n$, H is the join of the graphs $H[\{u_j : j \in J_1\}]$ and $H[\{u_j : j \in J_2\}]$. This contradicts the assumption that H is indecomposable. □

Lemma 2.10 *Let* $G^{::}$ *be a multiplication of a graph* G. *If* w, w^* *are two distinct vertices belonging to the same multivertex* W *of* $G^{::}$, *then there exists a homomorphism* $\psi : G \longrightarrow G - w^*$.

Proof According to the definition of multiplications of graphs, the neighbourhoods $N_{G^{::}}(w)$ and $N_{G^{::}}(w^*)$ are identical and $\{w, w^*\} \notin E(G^{::})$. Therefore, the mapping $\psi : G \longrightarrow G - w^*$ defined by

$$\psi(v) = \begin{cases} w & \text{for } v = w^*, \\ v & \text{otherwise.} \end{cases}$$

is a homomorphism of G to $G - w^*$. □

Corollary 2.11 *No multiplication* $H^{::}$ *is a core, besides the trivial case* $H^{::} = H^{::}(1) = H = C(H)$.

The multiplication operation strongly copies the structure of the original graph H. This can be expressed in the language of uniquely H-colourable graphs. This concept was introduced in [12]. We say that a graph G is *uniquely* H*-colourable* if there is a surjective homomorphism φ from G to H, such that any other homomorphism from G to H is the composition $\varphi \circ \alpha$ of φ and an automorphism α of H.

Theorem 2.12 *Let* H *be a core. Then any multiplication* $H^{::}(W_1, W_2, \ldots, W_n)$ *of* H *is uniquely* H*-colourable.*

Proof Let $v_1, v_2, \ldots, v_n$ be the vertices of H and let $W_1, W_2, \ldots, W_n$ be the corresponding multivertices of $H^{::}$.

For each $i = 1, 2, \ldots, n$, choose a vertex u_i of the multivertex W_i. It follows that the subgraph $H^{::}[\{u_1, u_2, \ldots, u_n\}]$ of $H^{::}$ is isomorphic to H. Define homomorphisms $\pi : H \longrightarrow H^{::}$ and $\varphi : H^{::} \longrightarrow H$ by

$$\pi(v_i) = u_i, i = 1, 2, \ldots, n,$$

$$\varphi(u) = v_i \text{ iff } u \in W_i, i = 1, 2, \ldots, n.$$

For any homomorphism $\psi : H^{::} \longrightarrow H$, $\pi \circ \psi$ is an endomorphism of H, and since H is a core, $\pi \circ \psi$ is an automorphism of H. In particular, ψ restricted to $H^{::}[u_1, u_2, \ldots, u_n]$ is surjective. Moreover, for any i, $1 \leq i \leq n$, and any $w \in W_i$, ψ restricted to the graph $H^{::}[\{u_1, u_2, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_n\}]$ is also surjective, and therefore $\psi(u_i) = \varphi(w)$. This means that in any homomorphism ψ, the whole multivertex W_i is mapped on the vertex $\psi(u_i)$ of H.

It follows that $\psi = \varphi \circ \pi \circ \psi$, i.e., every homomorphism $\psi : H^{::} \longrightarrow H$ is the composition of the homomorphism $\varphi : H^{::} \longrightarrow H$ and automorphism $\pi \circ \psi$ of H. □

3 Maximal graphs

Apart from being defined by the set of forbidden graphs, a hereditary property $\mathcal{P}$ can be also determined by the set of $\mathcal{P}$-maximal graphs. In this section we put stress on the latter type of characterization of $\mathcal{P}$. A graph is $\mathcal{P}$-*maximal* if $G + e \notin \mathcal{P}$ for any edge $e \in E(\overline{G})$. The class of all $\mathcal{P}$-maximal graphs is denoted by $\boldsymbol{M}(\mathcal{P})$.

Our aim is to describe the structure of $(\rightarrow H)$-maximal graphs. First we show that every $(\rightarrow H)$-maximal graph is a multiplication of a core.

Proposition 3.1 *Any $(\rightarrow H)$-maximal graph is a multiplication of a subgraph $\widetilde{H}$ of H, which is a core.*

5 A note on an extremal graph problem

The concept of maximal graphs with respect to hereditary properties is important also in connection with extremal graph theory. One of the most celebrated problems can be defined in the following way: Given a hereditary property $\mathcal{P}$, determine the maximum number of edges of the graphs on n vertices belonging to $\mathcal{P}$. This number is denoted by $\mathrm{ex}(n, \mathcal{P})$.

In order to describe the number $\mathrm{ex}(n, \mathcal{P})$, we introduce the concept of minimal forbidden graphs. As was already mentioned, any graph with hereditary property $\mathcal{P}$ is a subgraph of some $\mathcal{P}$-maximal graph. The other possible characterization of $\mathcal{P}$ is in terms of graphs not contained in $\mathcal{P}$. More precisely, we define the set

$$\mathbf{F}(\mathcal{P}) = \{F \in \mathcal{I} | F \notin \mathcal{P} \text{ but any proper subgraph } F^* \text{ of } F \text{ belongs to } \mathcal{P}\}.$$

Then a graph G belongs to $\mathcal{P}$ if and only if G contains no graph from $\mathbf{F}(\mathcal{P})$ as a subgraph. We further denote by $\chi(\mathcal{P})$ the number

$$\chi(\mathcal{P}) = \min\{\chi(F) | F \in \mathbf{F}(\mathcal{P})\}.$$

The following well-known result (see e.g. [11]) provides a connection between $\mathrm{ex}(n, \mathcal{P})$ and $\chi(\mathcal{P})$.

Theorem 5.1 *If $\mathcal{P}$ is a hereditary property with chromatic number $\chi(\mathcal{P})$ then*

$$ex(n, \mathcal{P}) = \left(1 - \frac{1}{\chi(\mathcal{P}) - 1}\right) \binom{n}{2} + o(n^2).$$

We point out that in the case of hom-properties we are seldom able to characterize the minimal forbidden graphs. But in spite of this, we are able to determine the value $\chi(\rightarrow H)$.

Proposition 5.2 *Let H be an arbitrary graph. Then $\chi(\rightarrow H) = \omega(H) + 1$.*

Proof As $G \subseteq G^{::}$ and $H \subseteq H^{::}$, we have $G + H \subseteq G^{::} + H^{::}$ and $G^{::} + H^{::}$ is a multiplication of $G + H$. It follows from Theorem 2.3 that $G + H$ is also a core, and therefore, by Corollary 3.5, $G^{::} + H^{::}$ is $(\to G_\circ \to H)$-maximal. □

4 A note on the computational complexity

In this section we remark on the computational complexity of recognizing graphs having hom-properties. The following nontrivial result is well known:

Theorem 4.1 [5] *For any fixed nonbipartite graph H, it is NP-complete to decide if a given graph G is homomorphic to H.*

It is slightly surprising that though the definition of maximal graphs involves general quantifiers, hom-maximal graphs are actually easier to recognize:

Theorem 4.2 *For every fixed graph H, it is polynomial to decide if a given graph G is $(\to H)$-maximal.*

Proof Given G, we first find $\widetilde{H}$ such that $G = \widetilde{H}^{::}$. This can be done in $O(n^3)$ time (where $n = |V(G)|$). If $\widetilde{H}$ is not a subgraph of H, we conclude that G is not maximal. Otherwise, the size of $\widetilde{H}$ is a constant. Then we check whether each homomorphism of $\widetilde{H}$ to H is injective and whether each subgraph of H isomorphic to $\widetilde{H}$ is induced (that is, we check whether $\widetilde{H}$ is itself $(\to H)$-maximal). This step takes only constant time. In the case of affirmative outcome, we check whether condition (iii) of Theorem 3.4 is satisfied, what can be also checked in constant time. Theorem 3.4 guarantees that this algorithm gives the correct answer. □

Proof Let G be a $(\to H)$-maximal graph. As $G \in \to H$, there is a homomorphism $\varphi' : G \to H$. Denote $\widetilde{H} = C(\varphi'(G))$. Since $\varphi'(G) \longrightarrow \widetilde{H}$, there is a homomorphism $\varphi : G \longrightarrow \widetilde{H}$. Denote by $v_1, v_2, \ldots, v_n$ the vertices of $\widetilde{H}$ and set $W_i = \varphi^{-1}(v_i)$. Then $G \subseteq \widetilde{H}^{::}(W_1, W_2, \ldots, W_n)$.

If there were $x \in W_i, y \in W_j$ such that $\{v_i, v_j\} \in E(\widetilde{H})$ and $\{x, y\} \notin E(G)$, the same φ would induce a homomorphism of $G + \{x, y\}$ to H, contradicting the assumption of maximality of G. Hence $G = \widetilde{H}^{::}(W_1, W_2, \ldots, W_n)$. □

It is not true in general that a multiplication of a core $\widetilde{H}$, which is a subgraph of the core H, is a $\to H$-maximal graph. The situation is more complicated, as illustrated by the following examples:

Example 3.2 The cycle C_5 has two cores as a subgraphs: C_5 itself and the complete graph K_2. Simultaneously, the cycle C_3 contains also two cores: C_3 and K_2.

The multiplications of K_2 are exactly all complete bipartite graphs. Addition of any edge to a complete bipartite graph $K_{m,n}$ creates a triangle and therefore $K_{m,n} \in \mathbf{M}(\to C_5)$ for all positive m, n. On the other hand, $K_{1,1} \in \mathbf{M}(\to C_3)$, but $K_{1,2} \notin \mathbf{M}(\to C_3)$ because $K_{1,2} \subseteq C_3 \in \mathbf{M}(\to C_3)$.

Lemma 3.3 *If a core $\widetilde{H}$ is $(\to H)$-maximal then every homomorphism from $\widetilde{H}$ to H is injective.*

Proof Let $\varphi : \widetilde{H} \longrightarrow H$ be a homomorphism such that $\varphi(u) = \varphi(v)$ for some $u \neq v$. Then $\{u, v\} \notin E(\widetilde{H})$ and it follows from the $(\to H)$-maximality of $\widetilde{H}$ that u and v have the same neighborhoods. Hence $\widetilde{H}$ is a multiplication of $\widetilde{H} - v$, contradicting Corollary 2.11. □

The following theorem characterizes which multiplications are hom-maximal.

Theorem 3.4 *A graph G is $(\to H)$-maximal if and only if G is a multiplication of a graph $\widetilde{H} \subseteq H$ (say $G = \widetilde{H}^{::}(W_1, \ldots, W_n)$ with W_i being the multivertex corresponding to a vertex v_i, $i = 1, 2, \ldots, n$) such that*

(i) $\widetilde{H}$ is a core;

(ii) $\widetilde{H}$ is $(\to H)$-maximal; and

(iii) $|W_i| = 1$ for every vertex $v_i \in V(\widetilde{H})$ for which there exists a homomorphism $\varphi : \widetilde{H} \longrightarrow H$ and a vertex $y \in V(H) - V(\varphi(\widetilde{H}))$ such that the closed $\varphi(\widetilde{H})$-neighborhood of $\varphi(v_i)$ is contained in the H-neighborhood of y.

Proof Suppose first that G is $(\to H)$-maximal. It follows from Proposition 3.1 that $G = \widetilde{H}^{::}(W_1, \ldots, W_n)$ for a core $\widetilde{H} \subseteq H$. Then $\widetilde{H}$ must itself be $(\to H)$-maximal, as a multiplication of a nonmaximal graph is nonmaximal as well. Let $\varphi : \widetilde{H} \longrightarrow H$ be a homomorphism such that $N_{\varphi(\widetilde{H})}(v_i) \cup \{v_i\} \subset N_H(y)$ for some $y \in V(H) - \varphi(\widetilde{H})$ and suppose $|W_i| > 1$, say $u \neq v \in W_i$. Then the mapping $\psi : G \longrightarrow H$ defined by

$$\psi(x) = \begin{cases} y & \text{for } x = u \\ \varphi(x) & \text{otherwise} \end{cases}$$

is a homomorphism of $G + \{u, v\}$ to H and G would not be $(\to H)$-maximal.

On the other hand, let $G = \widetilde{H}^{::}(W_1, \ldots, W_n)$ for a core $\widetilde{H} \subseteq H$, let $\widetilde{H}$ be $(\to H)$-maximal and let the condition (iii) be fulfilled. We will show that G is $(\to H)$-maximal. Suppose for the contrary that $G + \{u, v\} \in \to H$ for some vertices $u \neq v \in V(G)$, $\{u, v\} \notin E(G)$.

First, let u, v belong to different multivertices of the multiplication $\widetilde{H}^{::}$, say $u \in W_k$, $v \in W_l$, $k \neq l$. We choose vertices $u_i \in W_i$ for $i = 1, 2, \ldots, n$, so that $u_k = u, u_l = v$. Then $G[u_1, \ldots, u_n] \cong \widetilde{H}$, and it follows from $(\to H)$-maximality of $\widetilde{H}$ that $G[u_1, \ldots, u_n] + \{u_k, u_l\} \notin \to H$. Thus $G + \{u, v\} \notin \to H$ in this case.

Suppose now that u, v belong to the same multivertex, say W_i. Let $\psi : G + \{u, v\} \longrightarrow H$ be a homomorphism. Choose again vertices $u_j \in W_j$, $j = 1, 2, \ldots, n$ (so that $u_i = u$) and consider $\widetilde{G} = G[u_1, \ldots, u_n] \cong \widetilde{H}$. Denote by φ the homomorphism from $\widetilde{H}$ to H which maps v_i onto $\psi(u_i)$, $i = 1, 2, \ldots, n$. We claim that $\psi(v) \notin \varphi(\widetilde{H})$: Since $\{u, v\} \in E(G + \{u, v\})$, $\varphi(v_i) = \psi(u_i)$ is adjacent to $\psi(v)$ in H, and consequently $\psi(v) \neq \varphi(v_i)$. If $\psi(v) = \varphi(v_j)$ for some $j \neq i$, then $\{\varphi(v_i), \varphi(v_j)\} \in E(H)$ and (since $\widetilde{H}$ is $(\to H)$-maximal) $\{v_i, v_j\} \in E(\widetilde{H})$. Therefore, $\{v, u_j\} \in E(G)$ (since $v, u_i \in W_i$ and $u_j \in W_j$), contradicting $\psi(v) = \psi(u_j)(=\varphi(v_j))$.

The vertices u and v have the same neighborhood in G, and hence $N_{\varphi(\widetilde{H})}(v_i) \subset N_H(\psi(v))$. Thus $\psi(v) \in V(H) - V(\varphi(\widetilde{H}))$ plays the role of the bad guy y from condition (iii) ($|W_i| \geq 2$ is assumed and we have shown above that $\{\varphi(v_i), \psi(v)\} \in E(H)$). $\square$

Corollary 3.5 *Let $\to H$ be a hom-property. Then any multiplication $H^{::}(W_1, W_2, \ldots, W_n)$ of the core H is a $\to H$-maximal graph.*

Proof By the assumption, $\widetilde{H} = H$ is a core and hence also a $(\to H)$-maximal graph. Condition (iii) trivially holds, because $V(H) - V(\varphi(H)) = \emptyset$ for any homomorphism $\varphi : H \longrightarrow H$. $\square$

It was proved in [7] that reducible hom-properties are exactly products of hom-properties. Graphs maximal with respect to reducible hereditary properties were investigated in [2]. There it was proved that every $\mathcal{P}_1 \circ \mathcal{P}_2$-graph is the join of some $\mathcal{P}_1$-maximal graph and some $\mathcal{P}_2$-maximal one and the opposite implication is not valid in general. It follows from Theorem 3.4, that neither the join of maximal graphs with respect to hom-properties has to be maximal with respect to the join of these properties. The next result provides one type of sufficient conditions.

Corollary 3.6 *If $G^{::}$ is a multiplication of a core G and $H^{::}$ is a multiplication of a core H, then $G^{::} + H^{::}$ belongs to $\mathbf{M}(\to G \circ \to H)$.*