Some Optimization Problems on Solubility Sets of Separable Max-Min Equations and Inequalities

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1 Introduction

The aim of this paper is to suggest a direct Parametric method for solving some optimization problems on attainable sets of so called max-separable operators. Such problems in a less general form connected with the fuzzy set theory were considered e.g. in [1], [4]. The problem considered in this paper is presented independently of the fuzzy sets context as a non-linear nonconvex optimization problem. Parametic approach to its solution suggested is flexible enough to allow further extension and generalization, which are briefly discussed in the concluding sections.

2 Notations and Formulation of the Basic Problem

In this paper, we shall consider the following system of equations and inequalities

$$R_i(x) \equiv \max_{j \in N}(a_{ij} \wedge r_{ij}(x_j)) = b_i, \quad \forall i : i \in S$$

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$$h_j \le x_j \le H_j, \quad \forall j : j \in N \tag{1}$$

where $N \equiv \{1, 2, \dots n\}$, $S \equiv \{1, 2, \dots m\}$, $x \equiv (x_1, \dots x_n)^T \in R^n$, $b \equiv (b_1, \dots b_m)^T \in R^m$, $h \equiv (h_1, \dots h_n)^T \in R^n$, $H \equiv (H_1, \dots H_n)^T \in R^n$, $a_{ij} \wedge r_{ij}(x_j) \equiv \min(a_{ij}, r_{ij}(x_j))$, $R(x) = (R_1(x), \dots, R_m(x))^T$, let us assume further that $r_{ij} : R \to R$ are given strictly increasing continuous functions $\forall i : i \in S$, $\forall j : j \in N$. Using the above vector notation we can reformulate the system(1) as follows:

$$R(x) = b, \qquad h \le x \le H \tag{2}$$

Denote the set of all solutions of the system (1) (or (2)) by $M(b)$. Each component of $R : E^n \to E^m$ is a function depending on n variables; this function is expressed as the maximum n nondecreasing functions of one variable of the form $a_{ij} \wedge r_{ij}(x_j)$, so that these functions are separated by a max-operation. By similarity with the additive separability, we call this property of the functions $R_i(x)$ max-separablity and $R(x)$ is called a max-separable operator.

The vector b in the system (2) can be understood as a vector, which can be attained by an appropriate choice of $x \in M(b)$. Therefore those b's, for which $M(b) \neq \emptyset$, are called attainable elements and the set

$$A \equiv \{b / M(b) \neq \emptyset\} \tag{3}$$

is called the attainable set.

If an element $\hat{b} \in A$, then there exists a solution of the system (2) with $b = \hat{b}$ which can be obtained using some of the methods described in the literature (see e.g. [2] [3]). If $\hat{b} \notin A$, we want to find an approximate solution of the system (2) with the right hand side $\hat{b}$. For this purpose, we look for an element $b^{opt} \in A$, which has in some sense the minimal distance from $\hat{b}$ and accept the elements of $M(b^{opt})$ as appropriate approximate solutions.

In this article, we shall use the Tshebyshev distance, i.e. the following distance:

$$\| b - \hat{b} \| \equiv \max_{i \in S} |b_i - \hat{b}_i| \tag{4}$$

The problem, we are going to solve here is thus in the following form:

$$\| b - \hat{b} \| \to \min \quad \text{subject to} \quad b \in A \tag{5}$$

[5] Zimmermann K.: On Some Extremal Optimization Problems. Ekonomicko-matematický obzor, 1979, No.4.

[6] Zimmermann K.: Solution of Some Optimization Problems on Extremal Algebra, Methods in OR, Studies in Math. Programming (Ed. A.Prékopa) Akademiai Kiadó, Budapest 1980, pp.179-186.

[7] Zimmermann K.: The explicit solution of max-separable optimization problem, Ekonomicko-matematický obzor, 1982, No.4

[8] Zimmermann K.: On max-separable optimization problems. Annals of Discrete Mathematics 19, 1984, North Holland.

Since if $b \in A$, it means that there exists x such that $b = R(x)$ so that we can reformulate the problem (5) as follows:

$$\| R(x) - \hat{b} \| \equiv \max_{i \in S} |R_i(x) - \hat{b}_i| \to \min \quad \text{subject to} \quad h \leq x \leq H \tag{6}$$

The reformulation (6) shows that we minimize a continuous function of x on a compact set, so that there exists always at least one optimal solution x^{opt} of the system (6); thus if we set $b^{opt} \equiv R(x^{opt})$, we will obtain an optimal solution of the problem (5).

Let us define the set $M(t)$ for any t: $t \in [0, \infty)$ as follows:

$$M(t) \equiv \{x / h \leq x \leq H \ \& \ \| R(x) - \hat{b} \| \leq t\}. \tag{7}$$

The set $M(t)$ is nonempty if and only if the following system of inequalities:

$$R_i(x) \leq \hat{b}_i + t, \quad i \in S \ \& \ R_i(x) \geq \hat{b}_i - t, \quad i \in S \ \& \ h \leq x \leq H, \tag{8}$$

is soluble with respect to x; note that the set $M(t)$ is the set of all solutions x of (8). We can replace our original problems (5), (6) by the following problem:

$$t \to \min \quad \text{subject to} \quad M(t) \neq \emptyset \tag{9}$$

We shall show in the sequel that there exists always the optimal solution $t^{opt} \geq 0$ of the problem(9) and also we will derive a direct numerical procedure for determining t^{opt}. If x^{opt} is an any element of $M(t^{opt})$, then $\| R(x^{opt}) - \hat{b} \| \leq t^{opt}$; since the strict inequality can't hold, the equality must occur i.e. $b^{opt} \equiv R(x^{opt}) \in A$ is the optimal solution of the problem (5), and the vector x^{opt} can be accepted as an approximate solution of the system (2) in the case that $\hat{b} \notin A$, since for any solution x of the system (2), we have $\| R(x) - \hat{b} \| \geq t^{opt}$.

In the next section we investigate some properties of the set $M(t)$ where $t \in [0, \infty)$. This will enable us to derive the direct solution method for the system (5).

3 Properties of $M(t)$

We shall introduce the following notations $\forall i : i \in S$, $\forall j : i \in N$, $t \in [0, \infty)$:

$$\begin{aligned} V_{ij}(t) &= \{x_j / h_j \leq x_j \leq H_j \quad \text{and} \quad a_{ij} \wedge r_{ij}(x_j) \leq \hat{b}_i + t\} \\ V_j(t) &= \bigcap_{i \in S} V_{ij}(t) \ \& \ W_{ij}(t) = \{x_j / h_j \leq x_j \leq H_j \quad \text{and} \quad a_{ij} \wedge r_{ij}(x_j) \geq \hat{b}_i - t\} \end{aligned}$$

For the illustration of these sets see the Appendix. The following theorem gives the necessary and sufficient conditions for $M(t) \neq \emptyset$.

Theorem 3.1

$$M(t) \neq \emptyset \Leftrightarrow \left\{ \begin{array}{l} 1)\ V_j(t) \neq \emptyset, \ \ \forall j \in N \\ 2) \forall i \in S \ \exists j(i) \in N \ \text{ such that } \ W_{ij(i)}(t) \cap V_{j(i)}(t) \neq \emptyset. \end{array} \right\}$$

Proof: Define the interval $I_j = [h_j, H_j]$ and introduce the following notations:

$$\begin{aligned} & a1_{ij} = a_{ij} - (\hat{b}_i + t), \ \ a2_{ij} = a_{ij} - (\hat{b}_i - t) \ \ \& \ \ r1_{ij}(x_j) = r_{ij}(x_j) - (\hat{b}_i + t), \\ & r2_{ij}(x_j) = r_{ij}(x_j) - (\hat{b}_i - t) \end{aligned}$$

Sufficiency:
Assume that we have a point $x \equiv (x_1, \ldots, x_n)$ which satisfies the R.H.S. of the $\Leftrightarrow$-relation.

$$\begin{aligned} x_j \in V_j(t), \ \ \forall j \in N \ \ &\rightarrow \ \ x_j \in \bigcap_{i \in S} V_{ij}(t), \ \ \forall j \in N \\ &\rightarrow \ \ (a1_{ij} \wedge r1_{ij}(x_j) \leq 0, \ \ \forall i \in S; x_j \in I_j), \ \ \forall j \in N \\ &\rightarrow \ \ \max_{j \in N}(a1_{ij} \wedge r1_{ij}(x_j) \leq 0), \ \ \forall i \in S; \ \ \forall x_j \in I_j, \\ x_{j(i)} \in W_{ij(i)}(t) \ \ &\rightarrow \ \ a2_{ij(i)} \wedge r2_{ij(i)}(x_j) \geq 0; \ \ x_{j(i)} \in I_{j(i)} \\ &\rightarrow \ \ \max_{j \in N}(a2_{ij} \wedge r2_{ij}(x_j) \geq 0; \ \ x_{j(i)} \in I_{j(i)}. \end{aligned}$$

Then we can deduce that $x \in M(t)$.

$$1 \leq x_1 \leq 5, \quad 2 \leq x_2 \leq 6, \quad 0 \leq x_3 \leq 4, \quad 1 \leq x_4 \leq 3.$$

Then

$$V_1 = [1,5] \quad V_2 = \{2\} \quad V_3 = [0,4] \quad V_4 = [1,3],$$

and

$$V_1 \cap W_{11} \neq \emptyset, \quad V_2 \cap W_{22} \neq \emptyset, \quad V_3 \cap W_{33} \neq \emptyset,$$

where

$$W_{11} = [1,5] \quad W_{22} = [2,6] \quad W_{33} = [0,4]$$

then choose any x such that

$$x = (x_1, 2, x_3, x_4)$$

where

$$x_1 \in [1,5] \quad x_3 \in [0,4] \quad x_4 \in [1,3]$$

will be an approximate solution for the original problem.

References

[1] Cechlárová K., Cunninghame-Green R.A.: Residuation in Fuzzy Algebra and some Applications, Preprint No.93/22, Yhe University of Birmingham.

[2] Cunninghame-Green, R.A.: Minimax Algebra, Lecture Notes in Economics and Mathematical systems, Springer Verlag, 1979, 166.

[3] Jajou A., Zimmermann K.: Max-Separable Optimization Problems with Parameters in the Right-Hand Sides of the Constraints, 1985.

[4] Pedrycz W.: Inverse Problem in Fuzzy Relational Equations, Fuzzy Sets and Systems 36(1990), pp.277-291.

at $k = 3$

$$\begin{bmatrix} \xi_{131} & \xi_{132} & \xi_{133} & \xi_{134} \\ \xi_{231} & \xi_{232} & \xi_{233} & \xi_{234} \\ \xi_{331} & \xi_{332} & \xi_{333} & \xi_{334} \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} 0 & 9 & 1 & 3 \\ 0 \text{ if } 0 \le t < 1 & \begin{cases} 0 & \text{if } 1 < t \le \frac{3}{2} \\ 3 & \text{otherwise} \end{cases} & \begin{cases} -1 & \text{if } 2 < t \le \frac{10}{3} \\ 6 & \text{otherwise} \end{cases} & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

From the definition of ξ^{ij} given in (14), it is easy to obtain the following

$$\begin{bmatrix} \xi^{11} & \xi^{12} & \xi^{13} & \xi^{14} \\ \xi^{21} & \xi^{22} & \xi^{23} & \xi^{24} \\ \xi^{31} & \xi^{32} & \xi^{33} & \xi^{34} \end{bmatrix} \equiv \begin{bmatrix} 0 & 9 & 1 & 3 \\ 4 & \begin{cases} 10 & \text{if } 1 < t \le \frac{11}{2} \\ 3 & \text{otherwise} \end{cases} & 6 & 3 \\ 3/5 & \begin{cases} 10 & \text{if } 0 < t \le \frac{23}{8} \\ 0 & \text{otherwise} \end{cases} & 1 & 4 \end{bmatrix}$$

Hence t^{opt} which given by equation (15) will be

$$t^{opt} = \max(9, 3) = 9$$

Take any point (say x^*) from the set

$$M(b) = \{x : h \le x \le H, \quad \| R(x) - b \| \le t, \quad t \ge 9\},$$

then x^* will be accepted as an approximate solution of our problem.

In the original case $V_{12} = \emptyset$ {since $15 \wedge 7x_2 > 5,\ x_2 \in [2, 6]$},

$$V_{12} = \emptyset \to V_2 = \emptyset \longrightarrow M = \emptyset$$

i.e. there is no solution for the original problem.

In the modified case, if we take $t = 9$, then we try to solve the following problem

$$\begin{aligned} -4 &\le \max 7 \wedge 4x_1, 15 \wedge 7x_2, 4 \wedge 6x_3, 2 \wedge (x_4 + 1) \le 14 \\ -2 &\le \max 3 \wedge 2x_1, 4 \wedge x_2, 1 \wedge (x_3 + 1), 4 \wedge 2x_4 \le 16 \\ -6 &\le \max 3 \wedge (x_1 + 1), 3 \wedge (x_2 - 1), 2 \wedge (2x_3 - 1), -1 \wedge (x_4 - 3) \le 12 \end{aligned}$$

Necessity:

$$\begin{aligned} x \in M(t) \quad &\to \quad x_j \in I_j, \ \ \forall j \in N \ \ \& \ \ (\max_{j \in N} a1_{ij} \wedge r1_{ij}(x_j) \le 0 \ \ \forall i \in S) \\ &\qquad\qquad\qquad\qquad (\max_{j \in N} a2_{ij} \wedge r2_{ij}(x_j) \ge 0 \ \ \forall i \in S) \\ &\to \quad (x_j \in I_j, \ \ \text{and} \ \ a1_{ij} \wedge r1_{ij}(x_j) \le 0 \ \ \forall i \in S), \forall j \in N \\ &\qquad \& \forall i \in S \ \ \exists j(i) \in N \ \ \text{such that} \ \ a2_{ij(i)} \wedge r2_{ij(i)}(x_j) \ge 0; \ \ x_{j(i)} \in I_{j(i)} \\ &\to \quad 1) \ V_j(t) \ne \emptyset, \ \ \forall j \in N \\ &\qquad 2) \ \forall i \in S \ \ \exists j(i) \in N \ \ \text{such that} \ \ W_{ij(i)}(t) \cap V_{j(i)}(t) \ne \emptyset. \end{aligned}$$

Thus the proof of the theorem is complete. □

4 Properties of $V(t)$ & $W(t)$

We shall investigate here the conditions for $V_{ij}(t) \ne \emptyset$, $V_j(t) \ne \emptyset$, $W_{ij}(t) \ne \emptyset$. Define the following variables:

$$\begin{aligned} \eta^{ij} \ &\equiv \ \min(\max\{a_{ij} - \hat{b}_i, 0\}, \ \max\{r_{ij}(h_j) - \hat{b}_i, 0\}), \qquad \eta^j \equiv \max_{i \in S} \eta^{ij}, \\ \eta \ &\equiv \ \max_{j \in N} \eta^j \quad \text{and} \quad \tau^{ij} \equiv \max(0, \hat{b}_i - a_{ij}, \hat{b}_i - r_{ij}(H_j)). \end{aligned} \tag{10}$$

For the illustration of these variables see the appendix.

Theorem 4.1 *For each $j \in N$, $\exists \eta^{ij} \ge 0$ such that $V_{ij}(t) \ne \emptyset \Leftrightarrow t \ge \eta^{ij}$.*

Proof:

$$\begin{aligned} V_{ij}(t) = \emptyset \quad &\Leftrightarrow \quad a_{ij} \wedge r_{ij}(x_j) > \hat{b}_i + t; \qquad \forall x_j \in I_j \\ &\Leftrightarrow \quad \hat{b}_i + t < r_{ij}(x_j) < a_{ij} \text{ or } \hat{b}_i + t < a_{ij} < r_{ij}(x_j); \text{ it is further } t \geq 0 \\ &\Leftrightarrow \quad t < \min(\max\{r_{ij}(h_j) - \hat{b}_i, 0\}, \max\{a_{ij} - \hat{b}_i, 0\}) = \eta^{ij} \end{aligned}$$

where η^{ij} is given by (10); this completes the proof of the theorem. □

Theorem 4.2 *For each $j \in N$, $V_j(t) \neq \emptyset \Leftrightarrow t \geq \eta^j$; where η^j is given by (10).*

Proof: $V_j(t) = \emptyset$ equivalent to the fact $V_{i_0 j}(t) = \emptyset$ for some $i_0 \in S$; since V_{ij_0} are nested[1] sets *for fixed* $j_0 \in N$; which means that $t < \eta^{i_0 j} \leq \eta^j$ which is the maximum of η^{ij} on S. □

Corollary 4.1 *For each $j \in N$, $V_j(t) \neq \emptyset \Leftrightarrow t \geq \eta$; where η is given by (10).*

Proof: The proof is obviously derived from theorem 4.2.

Corollary 4.2 $M(t) \neq \emptyset \Leftrightarrow t \geq \eta$

Proof: The proof is obviously derived from theorem 3.1, theorem 4.2 and corollary 4.1; where η *is given by* (10).

Theorem 4.3 *For each $i \in S$, $j \in N$; $\exists \tau^{ij}$ such that $W_{ij}(t) \neq \emptyset \Leftrightarrow t \geq \tau^{ij}$.*

Proof:

$$\begin{aligned} W_{ij}(t) = \emptyset \quad &\Leftrightarrow \quad a_{ij} \wedge r_{ij}(x_j) < \hat{b}_i - t; \qquad \forall x_j \in I_j \\ &\Leftrightarrow \quad a_{ij} < r_{ij}(x_j) < \hat{b}_i - t \text{ or } r_{ij}(x_j) < a_{ij} < \hat{b}_i - t; \text{ it is further } t \geq 0 \\ &\Leftrightarrow \quad t < -\min(0, a_{ij} - \hat{b}_i, r_{ij}(H_j) - \hat{b}_i) \\ &\Leftrightarrow \quad t < \max(0, \hat{b}_i - a_{ij}, \hat{b}_i) r_{ij}(H_j)) = \tau^{ij} \end{aligned}$$

where τ^{ij} is given by (10); this completes the proof of the theorem. □

[1]Since for each j, $1 \leq j \leq n$, there exists a permutation $\{i, \ldots, i_m\}$ such that $V_{i_1 j} \subset V_{i_2 j} \subset \ldots \subset V_{i_m j}$ because of the fact that $a_{ij} \wedge r_{ij}(x_j)$ are nondecreasing in x_j.

at $k = 3$

$$\begin{bmatrix} \eta_{131} & \eta_{132} & \eta_{133} & \eta_{134} \\ \eta_{231} & \eta_{232} & \eta_{233} & \eta_{234} \\ \eta_{331} & \eta_{332} & \eta_{333} & \eta_{334} \end{bmatrix} = \begin{bmatrix} \frac{-3}{5} & \frac{-23}{8} & \frac{-7}{4} & -1 \\ 1 & \frac{3}{2} & \frac{8}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now from equation (11) and the values of the above parameters, we can obtain the following matrices
at $k = 1$

$$\begin{bmatrix} \xi_{111} & \xi_{112} & \xi_{113} & \xi_{114} \\ \xi_{211} & \xi_{212} & \xi_{213} & \xi_{214} \\ \xi_{311} & \xi_{312} & \xi_{313} & \xi_{314} \end{bmatrix} \equiv$$

$$\equiv \begin{bmatrix} 0 & 9 & 1 & 3 \\ 2 \text{ if } 0 \leq t < 3 & \begin{cases} 10 & \text{if } 1 < t \leq \frac{11}{2} \\ 3 & \text{otherwise} \end{cases} & \begin{cases} -1 & \text{if } 2 < t \leq \frac{31}{7} \\ 6 & \text{otherwise} \end{cases} & 3 \\ 3/5 & \begin{cases} 10 & \text{if } 0 \leq t \leq \frac{23}{8} \\ 0 & \text{otherwise} \end{cases} & \begin{cases} -1 & \text{if } 0 < t \leq \frac{7}{4} \\ 1 & \text{otherwise} \end{cases} & 4 \end{bmatrix}$$

at $k = 2$

$$\begin{bmatrix} \xi_{121} & \xi_{122} & \xi_{123} & \xi_{124} \\ \xi_{221} & \xi_{222} & \xi_{223} & \xi_{224} \\ \xi_{321} & \xi_{322} & \xi_{323} & \xi_{324} \end{bmatrix} = \begin{bmatrix} 0 & 9 & 1 & 3 \\ 4 & 3 & 6 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 7 & 15 & 4 & 2 \\ 3 & 4 & 1 & 4 \\ 3 & 3 & 2 & -1 \end{bmatrix}$$

Note that this problem has no solution in general.
It is clear that $r_{ij}(h_j) \leq a_{ij} \leq r_{ij}(H_j)$ $\forall i \in S, j \in N$. Using the relations (10) we can deduce that

$$\begin{bmatrix} \eta^{11} & \eta^{12} & \eta^{13} & \eta^{14} \\ \eta^{21} & \eta^{22} & \eta^{23} & \eta^{24} \\ \eta^{31} & \eta^{32} & \eta^{33} & \eta^{34} \end{bmatrix} = \begin{bmatrix} 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

this gives that

$$\begin{bmatrix} \eta^{1} & \eta^{2} & \eta^{3} & \eta^{4} \end{bmatrix} = \begin{bmatrix} 0 & 9 & 0 & 0 \end{bmatrix}$$

which implies that $\eta \leq 9$.

Also we get from (10), the following

$$\begin{bmatrix} \tau^{11} & \tau^{12} & \tau^{13} & \tau^{14} \\ \tau^{21} & \tau^{22} & \tau^{23} & \tau^{24} \\ \tau^{31} & \tau^{32} & \tau^{33} & \tau^{34} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 3 \\ 4 & 3 & 6 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

From the equality

$$r_{kj}^{-1}(b_k + \eta_{ikj}) = r_{ij}^{-1}(b_i - \eta_{ikj})$$

We can calculate η_{ikj} for each $k \in S$; $i \in S$ and $j \in N$, then the application of the above relation will give us the following three matrices:
at $k = 1$

$$\begin{bmatrix} \eta_{111} & \eta_{112} & \eta_{113} & \eta_{114} \\ \eta_{211} & \eta_{212} & \eta_{213} & \eta_{214} \\ \eta_{311} & \eta_{312} & \eta_{313} & \eta_{314} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & \frac{11}{2} & \frac{31}{7} & \frac{-1}{3} \\ \frac{3}{5} & \frac{23}{8} & \frac{7}{4} & 1 \end{bmatrix}$$

at $k = 2$

$$\begin{bmatrix} \eta_{121} & \eta_{122} & \eta_{123} & \eta_{124} \\ \eta_{221} & \eta_{222} & \eta_{223} & \eta_{224} \\ \eta_{321} & \eta_{322} & \eta_{323} & \eta_{324} \end{bmatrix} = \begin{bmatrix} -3 & \frac{-11}{2} & \frac{-31}{7} & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ -1 & \frac{-3}{2} & \frac{-8}{3} & \frac{5}{3} \end{bmatrix}$$

Corollary 4.3 *For each $j \in N$, $i \in S$; $\exists \tau^{ij} \geq 0$ such that*
$\forall t : t < \max_{i \in S} \min_{j \in N} \max_{i \in S}(\eta, \tau^{ij}) \Rightarrow M(t) \neq \emptyset$

Proof: It is clear from corollary 4.1, 4.2 and theorem 3.1, where τ^{ij} is given by (10)

Let us define the following sets: $P_{ikj}(t) = W_{ij}(t) \cap V_{kj}(t)$; $\forall i, k \in S$, $j \in N$. To investigate the necessary and sufficient conditions for $P_{ikj}(t) \neq \emptyset$, assume that the variable η_{ikj} satisfies the following equation $r_{kj}^{-1}(\hat{b}_k + \eta_{ikj}) = r_{ij}^{-1}(\hat{b}_i - \eta_{ikj})$ for some $i, k \in S$, $j \in N$ and define the variables ξ_{ikj}, ζ_{ikj} and γ_{ikj} for some $i, k \in S$ and $j \in N$ as follows:

$$\xi_{ikj} = \begin{cases} \eta_{ikj} & \text{if } \max(\tau^{ij}, \eta^{ij}) < \eta_{ikj} < \min(a_{kj} - \hat{b}_k, \hat{b}_i - r_{ij}(h_j)) \\ a_{kj} - \hat{b}_k & \text{if } r_{kj}^{-1}(\hat{b}_k + t) \leq r_{ij}^{-1}(\hat{b}_i - t) < H_j \\ \max(\tau^{ij}, \eta^{ij}) & \text{otherwise} \end{cases} \quad (11)$$

$$\zeta_{ikj} = \begin{cases} \eta_{ikj} & \text{if } \max(\tau^{ij}, \eta^{ij}) < \eta_{ikj} < \min(r_{kj}(H_j) - \hat{b}_k, \hat{b}_i - r_{ij}(h_j)) \\ \max(\tau^{ij}, \eta^{ij}) & \text{otherwise} \end{cases} \quad (12)$$

$$\gamma_{ikj} = \max(\tau^{ij}, \eta^{ij}) \quad (13)$$

For the illustration of these variables see also the appendix.

Concerning the definition of the sets $V_{kj}(t)$ & $W_{ij}(t)$, if we assume that

$$r_{ij}(h_j) \leq a_{ij} \leq r_{ij}(H_j) \quad \text{for all} \quad i \in S, \; j \in N$$

then it is easy to recognize the following remarks:

Remark 1:

If the two sets $[r_{kj}(h_j) - \hat{b}_k, a_{kj} - \hat{b}_k]$ & $[\hat{b}_i - a_{ij}, \hat{b}_i - r_{ij}(h_j)]$ have an empty intersection, then we can deduce the following:

- if $a_{kj} - \hat{b}_k < \hat{b}_i - a_{ij} = \tau^{ij} = \max(\tau^{ij}, \eta^{ij})$, for some $k \in S$, then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < \hat{b}_i - a_{ij}$; and if $t \geq \hat{b}_i - a_{ij}$ the intersection equals $W_{ij}(t)$.

- if $\hat{b}_i - r_{ij}(h_j) < r_{kj}(h_j) - \hat{b}_k = \eta^{ij} = \max(\tau^{ij}, \eta^{ij})$, for some $k \in S$, then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < r_{kj}(h_j) - \hat{b}_k$; and if $t \geq r_{kj}(h_j) - \hat{b}_k$ the intersection equals $V_{kj}(t)$.

Remark 2:

If the two sets $[r_{kj}(h_j) - \hat{b}_k, a_{kj} - \hat{b}_k]$ & $[\hat{b}_i - a_{ij}, \hat{b}_i - r_{ij}(h_j)]$ have a single point in their intersection, then we can deduce the following:

- if the point of intersection is $x = \hat{b}_i - r_{ij}(h_j) = r_{kj}(h_j) - \hat{b}_k = \eta^{ij} = \max(\tau^{ij}, \eta^{ij})$, for some $k \in S$, then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < x$; and if $t \geq x$ the intersection equals $V_{kj}(t)$.

- if the point of intersection is $x = a_{kj} - \hat{b}_k = \hat{b}_i - a_{ij}$, for some $k \in S$, then we have the following two cases:
  $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < x$, given that $r_{ij}^{-1}(\hat{b}_i - t) \leq r_{kj}^{-1}(\hat{b}_k + t) < H_j$; and if $t \geq x$ the intersection equals $W_{ij}(t)$; $x = a_{kj} - \hat{b}_k$.
  $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < x$, given that $r_{kj}^{-1}(\hat{b}_k + t) \leq r_{ij}^{-1}(\hat{b}_i - t) < H_j$; and if $t \geq x$ the intersection equals $V_{kj}(t)$; $x = \hat{b}_i - a_{ij} = \tau^{ij} = \max(\tau^{ij}, \eta^{ij})$.

Remark 3:

Let

$$z_1 = r_{kj}(h_j) - \hat{b}_k \quad , \quad y_1 = a_{kj} - \hat{b}_k$$
$$z_2 = \hat{b}_i - a_{ij} \quad , \quad y_2 = \hat{b}_i - r_{ij}(h_j).$$

Assuming that $[z, y] = [z_1, y_1] \cap [z_2, y_2]$ one can found the following cases:

- if $[z, y] = [z_1, y_1]$, then there exits some t_0 such that

$$\tau^{ij} < \eta^{ij} = z_1 < t_0 < y_1 < y_2 \quad \text{and} \quad r_{kj}^{-1}(\hat{b}_k + t_0) = r_{ij}^{-1}(\hat{b}_i - t_0);$$

  i.e. $\max(\tau^{ij}, \eta^{ij}) < t_0 < \min(y_1, y_2)$,
  then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t \geq t_0$ the intersection is nonempty, (similarly the case $[z, y] = [z_2, y_2]$).

Step 2:
Calculate η_{ikj} from the equation

$$r_{kj}^{-1}(\hat{b}_k + \eta_{ikj}) = r_{kj}^{-1}(\hat{b}_i - \eta_{ikj})$$

for each $i, k \in S$ and each $j \in N$.

Step 3:
Find ξ_{ikj}, ζ_{ikj}, or γ_{ikj} from relations (11), (12), (13) for each $i, k \in S$ and each $j \in N$.

Step 4:
Find ξ^{ij}, ζ^{ij} or γ^{ij} from relations (14) for each $i \in S$ and $j \in N$.

Step 5:
Find t^{opt} from relations (16), (17), (18).

Example

Here we want to solve the following problem

$$R_i \equiv \max_{j \in N}(a_{ij} \wedge r_{ij}(x_j)) = b_i \quad \forall i \in S$$

and

$$h_j \leq x_j \leq H_j \quad \forall j \in N$$

where

$$N = \{1, 2, 3, 4\}; \quad S = \{1, 2, 3\}; \quad x = [x_1 x_2 x_3 x_4]^T;$$

$$b = [5\ 7\ 3]^T; \quad h = [1\ 2\ 0\ 1]^T; \quad H = [5\ 6\ 4\ 3]^T;$$

$$\begin{bmatrix} r_{11}(x_1) & r_{12}(x_2) & r_{13}(x_3) & r_{14}(x_4) \\ r_{21}(x_1) & r_{22}(x_2) & r_{23}(x_3) & r_{24}(x_4) \\ r_{31}(x_1) & r_{32}(x_2) & r_{33}(x_3) & r_{34}(x_4) \end{bmatrix} \equiv \begin{bmatrix} 4x_1 & 7x_2 & 6x_3 & x_4 + 1 \\ 2x_1 & x_2 & 6x_3 + 1 & 2x_4 \\ x_4 + 1 & x_2 - 1 & 2x_3 - 1 & x_4 - 3 \end{bmatrix}$$

and

$t \geq T^{ij}$ hold $\forall i : i \in S$, $\forall j : j \in N$; where T^{ij} is equal to one of the values ξ^{ij}, ζ^{ij} or γ^{ij} according to which of the conditions from Theorems 4.7, 4.8 and 4.9 are satisfied and t^j is the same as η^j which defined in (10). Then the optimal value of $t(t^{opt})$ calculated according to the following formula:

$$t^{opt} = \max(\max_{j \in N} t^j, \max_{i \in S} \min_{j \in N} T^{ij}). \tag{15}$$

Consequently we can deduce that, the optimal value of $t(t^{opt})$ is calculated according to the following theorem:

Theorem 4.10 *If t is the solution of problem (9), then t holds one of the following relations:*

$$\begin{aligned} \text{If } r_{ij}(h_j) &\leq a_{ij} \leq r_{ij}(H_j) \;\text{ for all }\; i \in S,\; j \in N;\; \text{then} \\ t &\geq t^{opt} = \max_{i \in S} \min_{j \in N} \xi^{ij}. \qquad (16) \\ \text{If } r_{ij}(h_j) &\leq r_{ij}(H_j) \leq a_{ij} \;\text{ for all }\; i \in S,\; j \in N;\; \text{then} \\ t &\geq t^{opt} = \max_{i \in S} \min_{j \in N} \zeta^{ij}. \qquad (17) \\ \text{If } a_{ij} &\leq r_{ij}(h_j) \leq r_{ij}(H_j) \;\text{ for all }\; i \in S,\; j \in N;\; \text{then} \\ t &\geq t^{opt} = \max_{i \in S} \min_{j \in N} \gamma^{ij}. \qquad (18) \end{aligned}$$

Where ξ^{ij}, ζ^{ij} and γ^{ij} are given in (14).

Proof: In our proof we will concentrate on the first case. Let $\xi^{i_0 j_0} = \max_{i \in S} \min_{j \in N} \xi^{ij}$, and assume that $t < \xi^{i_0 j_0}$, then from theorem 4.7 we can deduce that:

$$W_{i_0 j}(t) \cap V_j(t) = \emptyset; \quad \forall j \in N,$$

hence, according to theorem 3.1, $M(t) = \emptyset$; this complete the proof of the theorem. □

5 Algorithm for Calculating t^{opt}

Step 1:
Find η^{ij}, η^j, η and τ^{ij} from relations (10), for each $i \in S$ and each $j \in N$.

- if $[z, y] = [z_0, y_1]$, $z_0 = z_1 = z_2$ then there exits some t_0 such that

  $$\tau^{ij} = \eta^{ij} = z_0 < t_0 < y_1 < y_2 \quad \text{and} \quad r_{kj}^{-1}(\hat{b}_k + t_0) = r_{ij}^{-1}(\hat{b}_i - t_0);$$

  i.e. $\max(\tau^{ij}, \eta^{ij}) < t_0 < \min(y_1, y_2)$,
  then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < t_0$; and if $t \geq t_0$ the intersection is nonempty, (similarly the case $[z, y] = [z_0, y_2]$).

- if $[z, y] = [z_1, y_0]$, $y_0 = y_1 = y_2$ then there exits some t_0 such that

  $$\tau^{ij} < \eta^{ij} = z_1 < t_0 < y_0 \quad \text{and} \quad r_{kj}^{-1}(\hat{b}_k + t_0) = r_{ij}^{-1}(\hat{b}_i - t_0);$$

  i.e. $\max(\tau^{ij}, \eta^{ij}) < t_0 < \min(y_1, y_2)$,
  then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < t_0$; and if $t \geq t_0$ the intersection is nonempty, (similarly the case $[z, y] = [z_2, y_0]$).

- if $[z, y] = [z_1, y_2]$, then there exits some t_0 such that

  $$\tau^{ij} < \eta^{ij} = z_1 < t_0 < y_2 < y_1 \quad \text{and} \quad r_{kj}^{-1}(\hat{b}_k + t_0) = r_{ij}^{-1}(\hat{b}_i - t_0);$$

  i.e. $\max(\tau^{ij}, \eta^{ij}) < t_0 < \min(y_1, y_2)$,
  then $V_{kj}(t) \cap W_{ij}(t) = \emptyset$ if $t < t_0$; and if $t \geq t_0$ the intersection is nonempty, (similarly the case $[z, y] = [z_2, y_1]$).

Theorem 4.4 *Let $i, k \in S$, $j \in N$; $r_{ij}(h_j) \leq a_{ij} \leq r_{ij}(H_j)$ for all $i \in S$, $j \in N$; then $\exists \xi_{ikj} \geq 0$ such that $P_{ikj}(t) \neq \emptyset \Leftrightarrow t \geq \xi_{ikj}$.*

Proof: The proof is obviously derived from the above remarks and from the definition of ξ_{ikj} which is given in (11).

Theorem 4.5 *Let $i, k \in S$, $j \in N$; $r_{ij}(h_j) \leq r_{ij}(H_j) \leq a_{ij}$ for all $i \in S$, $j \in N$; then $\exists \zeta_{ikj} \geq 0$ such that $P_{ikj}(t) \neq \emptyset \Leftrightarrow t \geq \zeta_{ikj}$.*

Proof: The proof is obviously derived from the above remarks and from the definition of ζ_{ikj} which is given in (12).

Theorem 4.6 *Let $i, k \in S$, $j \in N$; $a_{ij} \leq r_{ij}(h_j) \leq r_{ij}(H_j)$ for all $i \in S$, $j \in N$; then $\exists \gamma_{ikj} \geq 0$ such that $P_{ikj}(t) \neq \emptyset \Leftrightarrow t \geq \gamma_{ikj}$.*

Proof: The proof is obviously derived from the above remarks and from the definition of γ_{ikj} which is given in (13).

To generalize the above three theorems we introduce the following lemmas and remarks.

Lemma 4.1 *Let $i, k \in S$, $j \in N$; then $\exists \delta^{ij} \geq 0$ such that $V_j(t) \cap W_{ij}(t) \neq \emptyset \Rightarrow t \geq \delta^{ij}$.*

Proof: Let $\delta^{ij} = \max(\tau^{ij}, \eta^j)$; assume that for some fixed $i \in S$, $j \in N$ (say i_0, j_0); $\eta^{j_0} \geq \tau^{i_0 j_0}$; then $t < \delta^{i_0 j_0} \Rightarrow t < \eta^{j_0} \Rightarrow V_{j_0}(t) \cap W_{i_0 j_0}(t) = \emptyset$ (th. 4.2). Similarly we can treat the other case, and then the proof is complete. □

Lemma 4.2 *Let $i, k \in S$, $j \in N$; then $\exists \beta \geq 0$ such that $V_j(t) \cap W_{ij}(t) \neq \emptyset \Rightarrow t \geq \beta$.*

Proof: Let $\beta = \max_{i \in S} \min_{j \in N} \delta^{ij}$.
Assume that $t < \beta \Rightarrow$ for some fixed $i \in S$, $j \in N$ (say i_0, j_0) we have $t < \delta^{i_0 j_0}$

$$\begin{aligned} &\rightarrow t < \max(\tau^{i_0 j_0}, \eta^{j_0}) \\ &\Rightarrow V_{j_0}(t) \cap W_{i_0 j_0}(t) = \emptyset \text{ (th. 4.1)}, \end{aligned}$$

and then the proof is complete. □

Remarks:

- From lemma 4.1 and lemma 4.2 we have
  $V_j(t) \neq \emptyset$ and $W_{ij}(t) \neq \emptyset$ if $t \geq \beta$ i.e. $t \in [\beta, \infty)$.
- If $t \geq \max(\tau^{ij}, \eta^j)$, then $V_j(t) \neq \emptyset$ and $W_{ij}(t) \neq \emptyset$.
- From th. 4.1 and th. 4.2 we have:
  $V_j(t) \neq \emptyset$ and $W_{ij}(t) \neq \emptyset$ if $t < \hat{b}_i - r_{ij}(h_j)$ and $t \leq \max(\tau^{ij}, \eta^j)$ i.e. $t \in [\max(\tau^{ij}, \eta^j), \hat{b}_i - r_{ij}(h_j))$.

If we redefine ξ_{ikj}, ζ_{ikj} and γ_{ikj} by replacing η^{ij} by η^j and then by η in (11), (12) (13), then from theorems 4.1 & 4.2, lemmas 4.1 & 4.2 and also from the above remarks, we can prove again the generalized form of theorems 4.4 & 4.5 & 4.6 which obtained by the new formulas of ξ_{ikj}, ζ_{ikj} and γ_{ikj}.

Now let us define the following maximum variables:

$$\xi^{ij} = \max_{k \in S} \xi_{ikj}, \quad \zeta^{ij} = \max_{k \in S} \zeta_{ikj}, \quad \text{and} \quad \gamma^{ij} = \max_{k \in S} \gamma_{ikj}. \tag{14}$$

The following three theorems give another sufficient and necessary conditions for

$$V_j(t) \cap W_{ij}(t) \neq \emptyset.$$

Theorem 4.7 *Let $i \in S$, $j \in N$; $r_{ij}(h_j) \leq a_{ij} \leq r_{ij}(H_j)$ for all $i \in S$, $j \in N$; then $\exists \xi^{ij} \geq 0$ such that $V_j(t) \cap W_{ij}(t) \neq \emptyset \Leftrightarrow t \geq \xi^{ij}$; ξ^{ij} is given by (14).*

Proof: The assertion follows immediately from theorem 4.4 and the definition of $V_j(t)$.

Theorem 4.8 *Let $i \in S$, $j \in N$; $r_{ij}(h_j) \leq r_{ij}(H_j) \leq a_{ij}$ for all $i \in S$, $j \in N$; then $\exists \zeta^{ij} \geq 0$ such that $V_j(t) \cap W_{ij}(t) \neq \emptyset \Leftrightarrow t \geq \zeta^{ij}$; ζ^{ij} is given by (14).*

Proof: The assertion follows immediately from theorem 4.5 and the definition of $V_j(t)$.

Theorem 4.9 *Let $i \in S$, $j \in N$; $a_{ij} \leq r_{ij}(h_j) \leq r_{ij}(H_j)$ for all $i \in S$, $j \in N$; then $\exists \gamma^{ij} \geq 0$ such that $V_j(t) \cap W_{ij}(t) \neq \emptyset \Leftrightarrow t \geq \gamma^{ij}$; γ^{ij} is given by (14).*

Proof: The assertion follows immediately from theorem 4.6 and the definition of $V_j(t)$.

¿From the above results we conclude that there exist some values, say t^j & T^{ij} for which the relations $V_j(t) \neq \emptyset \Leftrightarrow t \geq t^j$ and $W_{ij}(t) \cap V_j(t) \neq \emptyset \Leftrightarrow$