Fourth Prague Combinatorial Workshop

(edited by Martin Klazar)

July, 29 – August, 2, 1996

Content

Problems by

1 Preface

The Fourth Prague Midsummer Combinatorial Workshop was held from July 29 to August 2, 1996 at Malostranské náměstí building of Charles University whose one side is depicted by a drawing on the cover of these KAM Series. The workshop was organized by Department of Applied Mathematics (KAM) of Charles University and it was supported by DIMANET.

Only a small but distinguished group of mathematicians was invited and we were particularly happy to have Moshe Rosenfeld and Carsten Thomassen among us. It was important that graduate students could take an active part. This included not only our graduate students but as well graduate students from Universite de Paris and Mathematical Institute in Budapest. The workshop followed an informal daily routine which included morning and early afternoon discussions. This report reflects some of the discussions during the workshop. Perhaps you can digest from these proceedings some of the athmosphere at the workshop and you will also see that the fruitfull exchange of ideas led directly to some new results and papers.

This volume is edited by Martin Klazar. Most of the problems were supplied by the authors as TEX files, the rest in print form or as handwritings. In any case we appologize for any inaccuracies which might occur in the editing process.

The conference photos were taken in Ledebourg Gardens under Prague Castle on Wednesday, July 31. That day was perhaps the most enjoyable day: It started as a very scentifically active day in the morning. In the afternoon nearly everybody took part in the guided tour (J. N.) through newly opened Museum of Modern Art in Prague. In the early evening we had the conference photo and dinner in Ledebourg Gardens which was followed by an organ concert given in St. Nicolas Church by conference participant Andreas Huck (we include the concert programme in these proceedings). And after that nobody wanted to leave and we ended our day "U Kocoura". Well perhaps a really rich day in a true renaissance sense.

This workshop was partly supported by our University grant GAUK 194, our Czech grant GAČR 0194 and Discrete Mathematik Network (DIMANET). We thank to DIMANET coordinator Walter Deuber (Bielefeld).

Based on our past experience and beeing encouraged by several participants we hope to organize the Fifth Prague Midsummer Combinatorial Workshop from July 28 to August 1, 1997.

Jaroslav Nešetřil

2 Programme of the organ concert performed by Andreas Huck in St. Nicolas Church on July, 31

1. **Marcel Dupré** (1886 – 1971)
   Entrée

2. **Johann Sebastian Bach** (1685 – 1750)
   Prelude to the chorale *Jesus, meine Freude (Jesus, my delight)*
   from the *Orgelbüchlein (Little Organ Book)*

3. **Charles-Marie Widor** (1844 – 1937)
   Toccata of the 5th symphony in f-minor

4. **Jean Langlais** (1907 – 1991)
   Rosa Mystica from the *Triptyque Grégorian*
   (a contemporary arrangement of Gregorian chants)

5. **Johann Sebastian Bach** (1685 – 1750)
   Toccata and fugue in d-minor

3 Robert Babilon

Colorability of some classes of planar graphs

Motivation.

Let us have a face-to-face romb tiling (it means that near-by rombs have always common whole side) of the whole plane without holes and overlaps (see figure 1).

Let us consider an edge of any romb in the tiling. Then the tiling contains a strip (infinite in both directions) of pairwise adjancent rombs each of them having two edges parallel to the edge we started from (for example the filled circles in figure 1). We can put a curve inside every strip. If we erase the rombs we get a system of the (infinite) curves (see figure 2). We can easily show that every two curves have at most one intersection point and that no three (or more) curves are intersected in one point.

In this way we can define a (infinite) graph, vertices of which are the intersection points of the curves (corresponding to the rombs of the tiling) and edges connect near-by intersection points on the curves (corresponding to the adjancent rombs).

I proved that such a graph (and also face-to-face romb tiling) is 3-colorable ($\chi(G) \leq 3$). In every finite portion of the graph I found a vertex of degree at most two. The rest of the proof is easy if the mathematical induction and the diagonal method are used.

Problems.

1. Let us have a finite system of (finite) curves. Let no curve have selfintersection points. Let no three (or more) curves be intersected in one point. Let us define a graph like above (vertices are intersection points, edges connect near-by vertices on the curves). My question is whether every graph is 3-colorable in following three special cases:

(a) Ends of the curves are “outside” the graph, every two curves have at most one intersection point. This is the case described above (in Motivation) and already solved by me.

(b) Ends of the curves are “outside” the graph, but every two curves have arbitrary many intersection points (see figure 3). (If we get a multiedge, we can replace it by simple edge.)

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(c) Every two curves have at most one intersection point, but ends of the curves can be “inside” the graph (see figure 4).

In the last two cases ((b) and (c)) there exist graphs with all vertices of degrees at least three so we cannot use my argument.

2. Is every 3-regular planar graph (except K_4) 3-colorable?

4 Krystyna T. Balińska and Louis V. Quintas

Two problems on self-complementary graphs

4.1 Automorphisms

The self-complementary graphs (sc-graphs) on n vertices having maximum order automorphism group for each realizable n have been determined. A sc-graph with maximum order automorphism group is called *maximal.*

Theorem 4.1.1 *Let* $|Aut(G)|$ *denote the order of the automorphism group of a sc-graph* G*. Then,* $|Aut(G)| \leq a_n$*, where*

$$a_n = \begin{cases} 1 & if\; n = 1 \\ 10 & if\; n = 5 \\ 72 & if\; n = 9 \\ 2(p!)^4 & if\; n = 4p,\ with\ p \geq 1\ or\ n = 4p+1,\ with\ p \geq 3 \end{cases}$$

If $n = 4p$ *with* $p \geq 2$ *or if* $n = 4p + 1$ *with* $p \geq 3$*, then there are exactly two or four maximal graphs, respectively.* ■

A proof of this theorem was given by T. Luczak at the Discrete Mathematics Seminar, Adam Mickiewicz University, Poznań, May 14, 1996.

In Theorem 4.1.2 we use the following operation to describe the maximal graphs on n vertices.

Definition 4.1.1 *Let* X *be a labeled graph with vertex set* $\{1, 2, \ldots, s\}$ *with* $s \leq n$ *and* $H = (H_1, H_2, \ldots, H_s)$ *be a sequence of* s *disjoint labeled graphs such that the union of their vertex sets is* $\{1, 2, \ldots, n\}$*. Then,*

$$J(X; H) = \bigcup_{\{i,j\} \in E(X)} (H_i + H_j)$$

where + is the join operation, is called the graphical join *of* X *relative to* H *and results in a graph on vertex set* $\{1, 2, \ldots, n\}$*.*

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Note that the graphical join is a generalization of the sequential join operation $H_1 + H_2 + \ldots + H_s$ (see [1; p. 26]) which we can express as $J(P_s; H)$, where P_s is the path on s vertices.

Theorem 4.1.2 *The maximal sc-graphs G_n on n vertices are as follows.*

(a) $G_1 = K_1$, $G_4 = P_4$ (path on 4 vertices), $G_5 = C_5$ (cycle on 5 vertices), and $G_9 = L(K_{3,3})$ (line graph of $K_{3,3}$).

(b) If $n = 4p$ with $p \geq 2$, then

$$G_{4p}^{(1)} = J(P_4; H^{(1)}) \text{ and } G_{4p}^{(2)} = J(P_4; H^{(2)}),$$

where $H^{(1)} = (K_p^c, K_p, K_p, K_p^c)$ and $H^{(2)} = (K_p, K_p^c, K_p^c, K_p)$.

(c) If $n = 4p + 1$ with $p \geq 3$, then

$$G_{4p+1}^{(1)} = J(B_5; H^{(1)}, K_1), \quad G_{4p+1}^{(2)} = J(C_5; H^{(1)}, K_1),$$

$$G_{4p+1}^{(3)} = J(B_5; H^{(2)}, K_1), \text{ and } G_{4p+1}^{(4)} = J(C_5; H^{(2)}, K_1),$$

where B_5 is the sc-graph on 5 vertices – the triangle with twononincident pendant edges. ■

Drawings of maximal graphs are shown in the figure.

So problem 3 can also be reformulated in the language of hypertriangulations.

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in this anticlockwise order, then $\rho(V)$ is the permutation $\rho(V) = (\Delta_1, \Delta_2, \ldots, \Delta_p)$.

Let R denote the anticlockwise rotation of the labelled triangles around all r-vertices, i.e., if $V_1, V_2, \ldots, V_q$ are the vertices of colour r, then $R = \rho(V_1) * \rho(V_2) * \cdots * \rho(V_q)$. Let L and B denote the anticlockwise rotations of the labelled triangles around all l- and b-vertices, respectively. Besides trivial cases (R, L, B) is a regular triple in the symmetric group S_n on the n letters $1, 2, \ldots, n$. An example is presented in Fig. 4.

Next label the triangles of C_2 by the symbols $1', 2', \ldots, n'$. If R', B', L' are the anticlockwise rotations of the triangles of C_2 around the r-, b-, and l-vertices, respectively, then (R', B', L') is a regular triple in the symmetric group S_n on the letters $1', 2', \ldots, n'$.

Problem 2. How are $T(R, L, B)$ and $T(R', B', L')$ related? Are they isomorphic?

Let H be a vertex 3-colourable triangulation of a closed oriented surface. Then let $T(H)$ denote one of its two related Sachs traingulations $T(R, L, B)$ and $T(R', B', L')$.

If (x, y, z) is a regular triple of a finite group so that x, y, z are pairwise non-conjugated, then $T(x, y, z)$ is a vertex 3-colourable triangulation. $T(x, y, z)$ has precisely three pairwise distinct vertex orbits $O(x), O(y), O(z)$. These orbits define a vertex 3-colouring of $T(x, y, z)$.

Problem 3. Apply the construction above presented to $H = T(x, y, z)$. Which conditions imply that $T(H)$ is isomorphic to $T(x, y, z)$?

Remark 1. The octahedron $H = O_8$ is a Sachs triangulation (according Fig. 3 the octahedron $O_8 \simeq T((12)(34), (1234), (13))$. The Sachs triangulation $T(O_8)$ is a trivial one (see Fig. 4). Hence $T(O_8) \not\simeq O_8$.

Remark 2. The basic vertex 3-colourable triangulation H can also be considered as a vertex 3-colourable 3-uniform hypetriangulation H°, if the triangles of class C_1 are considered to be 3-edges of H°.

The automorphism groups of the maximal sc-graphs are as follows:

$Aut(K_1) \approx S_1$ (identity group),

$Aut(P_4) \approx S_2$ (symmetric group of order 2),

$Aut(C_5) \approx D_5$ (dihedral group of order 10), since $G_9 = L(K_{3,3})$,

$Aut(G_9) \approx Aut(L(K_{3,3})) \approx Aut(K_{3,3}) \approx Aut((K_{3,3})^c) \approx S_2[S_3]$ (wreath product of S_2 around S_3, which has order $2!(3!)^2 = 72$), and

$Aut(G_{4p}) \approx Aut(G_{4p+1}) \approx S_2 \times S_p \times S_p \times S_p \times S_p$ (direct product of S_2 and four copies of S_p, which has order $2(p!)^4$.

With respect to minimum orders note that other than K_1, the automorphism group of a sc-graph has even order and the minimum of 2 is realized for each $n = 4p, 4p+1$.

An algorithm [2] that calculates the order of the automorphism group of a graph has been applied to sc-graphs on up to 12 vertices [3,4] using the graphs listed by M. Kropar and R.C. Read [5].

The data generated [4] suggests that, in the set of sc-graphs on n vertices, the proportion of graphs having group order 2 increases with n. For example, if $n = 8$, then this proportion is $3/10$ whereas for $n = 12$ it is 528/720=11/15.

For a sc-graph on $4p$ or $4p + 1$ vertices, the *code* of its degree sequence, listed in non-decreasing order, consists of its first p terms having odd indices [5,6]. If sc-graphs are partitioned into classes in accordance with their codes, then certain classes have minimum group order greater than 2. For example, for $n = 12$ among the 20 classes there are exactly 6 such classes: '135', '144', '145', '155', '225', and '333' having minimum group order: 8, 16, 4, 4, 16, and 4 with the number of associated graphs: 1, 1, 2, 1, 1, and 1, respectively.

Problem 1

Let c be a code such that all sc-graphs with code c have automorphism group with order bigger than 2.

(a) For a given number of vertices, find all such codes c.

(b) For a given c, determine the minimum group order.

(c) For a given c, determine the number of sc-graphs with minimum group order.

4.2 Complete subgraphs

Let b_m with $m \geq 3$ denote the number of induced K_m subgraphs contained in a sc-graph on n vertices. A formula for the total number of triangles in a graph and its complement was derived by A.W. Goodman [7]. From this formula the number of triangles in a sc-graph follows directly.

Theorem 4.2.1 *Let $d_1, d_2, ..., d_n$ be the vertex degree sequence of a sc-graph on n vertices. Then,*

$$b_3 = (1/2)\binom{n}{3} - (1/4)\sum_{i=1}^{n} d_i d_i^c$$

where $d_i^c = n - 1 - d_i$. $\blacksquare$

The results on lower bounds (C.R.J. Clapham [8]) and upper bounds (S.B. Rao [9]) on b_3 as a function of the number of vertices n of a sc-graph can be expressed as the following theorem.

Theorem 4.2.2 *For a sc-graph on n vertices,*

$$\begin{array}{ll} n(n-2)(n-4)/48 \leq b_3 \leq n(n-4)(2n-1)/48 & \textit{if } n = 4p, \textit{ and} \\ n(n-1)(n-5)/48 \leq b_3 \leq (n-1)(2n^2-7n-3)/48 & \textit{if } n = 4p+1. \\ & \blacksquare \end{array}$$

For the case $m = \lfloor n/2 \rfloor$ we proved with S.B. Rao the following result.

Theorem 4.2.3 *If $m = \lfloor n/2 \rfloor$, then for sc-graphs on n vertices*

$$\begin{array}{ll} 0 \leq b_m \leq 3 & \textit{if } n = 4p, \textit{ and} \\ 0 \leq b_m \leq m+3 & \textit{if } n = 4p+1. \quad \blacksquare \end{array}$$

Theorem 4.2.4 *If G is a sc-graph on $n > 5$ vertices and $m = \lfloor n/2 \rfloor$, then the clique number $\omega(G)$ is bounded as follows.*

$$\begin{array}{ll} 3 \leq \omega(G) \leq m & \textit{if } n = 4p, \textit{ and} \\ 3 \leq \omega(G) \leq m+1 & \textit{if } n = 4p+1. \end{array}$$

Proof. A sc-graph G on $n > 5$ vertices cannot contain an induced K_s with $s > \lfloor n/2 \rfloor$, if n is even, nor with $s > \lfloor n/2 \rfloor + 1$ if n is

The general question presented by H. Sachs in 1963 [1] was the following:

Problem 1. How are properties of finite groups related to properties of their sets of Sachs triangulations and vice versa?

For further reading to this question the reader is referred to [2].

The following question has been dealt with at the Prague Midsummer Combinatorial Workshop in 1995 ([3]):

Which triangulations of closed oriented surfaces are Sachs triangulations?

17.2 Sachs triangulations generated by vertex 3-coloured triangualations of closed oriented surfaces

With the concept of Sachs triangulations 2-rotary maps or even regular maps with certain properties can be constructed. The problem is to find to a given property P a regular triple (x, y, z) of some finite group so that $T(x, y, z)$ has P. For these purposes maps D of closed oriented surfaces have been used. From D a regular triple (R, L, B) of permutations in some symmetric group and the according Sachs triangulation $T(D) = T(R, L, B)$ can be found. In general D is much simpler than $T(D)$. By a good choice of D a Sachs triangulation $T(R, L, B)$ with a given property can be obtained (see [4]). Here vertex 3-colourable triangulations H of closed oriented surfaces S will be used.

Let H be a triangulation of a closed oriented surface which has a vertex colouring by three colours — say r, l, b. Then H has two classes C_1 and C_2 of triangles; the triangles of class C_1 (and of class C_2) have the colours r, l, b (and r, b, l, respectively) on its bounding 3-cycle in this anticlockwise order.

Obviously, $|C_1| = |C_2|$. Let $n = |C_1| = |C_2|$. Label the triangles of class C_1 by $1, 2, \ldots, n$. An *anticlockwise rotation* $\rho(V)$ of the labelled triangles around a vertex V is defined as follows: if $\Delta_1, \Delta_2, \ldots, \Delta_p$ are the labelled triangles incident with V and occuring around V

odd. To see this, assume G contains such a K_s. Since G is self-complementary, G contains at least s independent vertices. These vertices are not vertices that induce the preceding K_s, thus they must be among the $n - s$ remaining vertices of G. However, if n is even, then $n - s < n - \lfloor n/2 \rfloor = \lfloor n/2 \rfloor$, a contradiction and if n is odd, then we have $n - s < n - \lfloor n/2 \rfloor - 1 = n - (n-1)/2 - 1 = (n-1)/2 = \lfloor n/2 \rfloor$, a contradiction. Therefore, $s \leq \lfloor n/2 \rfloor$ if n is even and $s \leq \lfloor n/2 \rfloor + 1$ if n is odd.
From Theorem 4.2.2 every sc-graph on $n > 5$ vertices contains a triangle. ■

Problem 2
(a) Determine the lower and upper bounds for b_m *when* $3 < m < \lfloor n/2 \rfloor$.
(b) Determine the number of sc-graphs on n *vertices having the largest induced* K_m.
(c) Determine the number of sc-graphs on n *vertices having the maximum* b_m.

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5 Olof Barr

Long cycles in the middle two levels of the Boolean lattice

The Hasse diagram of the Boolean lattice is a wellstudied object in mathematics with many an implication for datastructures and computer algorithms. Here we present some problems concerning the existence of long cycles or paths in special subgraphs of this Hasse diagram.

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is in an triangle of Σ, too. In the sense of combinatorial topology opposite directed arcs (x, y), (y, x) are identified so that an edge $[x, y]$ is obtained (Fig. 2). Identifying step by step all opposite directed arcs a set Θ of triangulations of closed oriented surface is obtained.

If the elements x, y of the regular triple (x, y, z) commute, then $xyz = yxz = e$ and the triangulation T is formed by the two triangles $D(x, y, z)$ and $D(y, x, z)$ only. Hence T is the sphere with a cycle of length 3 on it which triangulates the sphere. It is called the *trivial triangulation*.

Consequently, in the following we shall consider *non-commuting* elements of G. Really, two non-commuting elements x, y form with $z = (xy)^{-1}$ a regular triple (x, y, z). Thus to each pair of non-commuting elements $x, y \in G$ a triangulation $T(x, y, z)$ of a closed oriented surface is obtained. Such a triangulation is called a *Sachs triangulation*. It is a component of the set Θ defined above. The octahedron $T((12)(34), (1234), (13))$ is presented as an example in Fig. 3. It belongs to the symmetric group S_4 on the four letters 1, 2, 3, 4.

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17 Heinz-Juergen Voss

In 1963 H. Sachs [1] developed the concept of describing finite groups by triangulations of closed oriented surfaces. Since then this description has been studied by W. Voss and H.-J. Voss — for a survey see H.-J. Voss [2]. The problems here presented are related to this description.

17.1 Concepts

Let G be a finite group. A *triple* (x, y, z) of elements $x, y, z \in G$ is said to be *regular*, if $x \neq y \neq z \neq x$ and $xyz = e$. With $xyz = e$ also $yzx = e$ and $zxy = e$. If (x, y, z) is a regular triple, then (y, x, z') with $z' = (yx)^{-1}$ is regular, too. To each regular triple (x, y, z) an *oriented triangle* $D(x, y, z)$ with arcs (x, y), (y, z), (z, x) and vertices x, y, z is assigned so that $D(x, y, z) = D(x', y', z')$ iff $(x', y', z') \in \{(x, y, z), (y, z, x), (z, x, y)\}$; otherwise $D(x, y, z)$ and $D(x', y', z')$ are disjoint (Fig. 1). In the set Σ of all triangles so obtained an arc occurs at most once. If (x, y) is in some triangle of Σ, then (y, x)

Let $\mathcal{B}_n$ be the n-atom Boolean lattice, i.e. the partially ordered set of subsets of the set $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion. Let $H(\mathcal{B}_n) = (V_n, E_n)$ be the Hasse diagram, or the covering graph, of $\mathcal{B}_n$, so that the vertices V_n are the subsets of $[n]$ with two subsets adjacent when they differ in just one element.

The first conjecture has variously been attributed to Hável, Dejter, Erdős, Trotter and Kelley.

Conjecture 1 (middle levels) *There is a Hamilton cycle (or path) in the two layers of the Hasse diagram (on the Boolean lattice) representing all sets of order k and $k+1$ respectively if $n = 2k+1$.*

It has been shown in [15] by Savage and Winkler that you always can find a cycle covering 83% of the vertices using monotone Gray codes.

The conjecture above can be generalized to the following question about adjacent levels in the Hasse diagram:

Conjecture 2 (adjacent levels) *There is a cycle (or path) in two adjacent levels of the Hasse diagram (on the Boolean lattice) that covers the smaller part of the bipartite graph induced on the vertices representing all sets of order r and $r+1$.*

The following connection is easy to establish:

Proposition 1 *If Conjecture 1 is true, then is Conjecture 2 true.*

The conjecture about the middle levels having a Hamilton path is a special instance of an old conjecture of Lovász about vertex-transitive graphs.

Conjecture 3 (Lovász) *Every vertex-transitive graph has a Hamilton path.*

So far is the longest known path of size $C\sqrt{n}$, a result due to Babai in [1].

With the known information about the middle levels we can make weaker conjectures of the form:

Conjecture 4 *Every vertex-transitive graph with property P has a Hamilton path, where P could be bipartite, edge-transitive or any other combination of properties of the middle levels in the above mentioned Hasse diagram.*

The above mentioned problems are also connected with the following conjecture.

Conjecture 5 *Every odd graph has a Hamilton path.*

Ivan Havel has shown in [9] that for special cases of n is Conjecture 1 implied by Conjecture 5. It is also conjectured that Conjecture 5 implies Conjecture 1 in all well-formulated cases.

To make it easier for the interested reader an extended list of references is included below.

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(and similarly for $a'(X)$). Then the conjecture of Aharoni and Keich can be stated as follows:

Conjecture 4 *Suppose that for all possible types κ we have that*

$$\max\{a(X) : X \textit{ is of type } \kappa\} \geq \max\{a'(X) : X \textit{ is of type } \kappa\}.$$

Then it follows that

$$\sum_{X \textit{ is of type } \kappa} a(X) \geq \sum_{X \textit{ is of type } \kappa} a'(X).$$

Suppose that A and A' are $0-1$-matrices and denote by A_i (resp. A'_i) the set whose characteristic vector is the i-th row of A (resp. A'). The product $a(X)$ is either zero or one, and it is one if and only if $X_i \subset A_i$ for all i. In this latter case, we call the partition X *compatible* with the set system $(A_1, \ldots, A_n)$. The Aharoni-Keich-Conjecture for $0-1$-matrices can now be stated as follows:

Conjecture 5 *Suppose that, for each type κ, there is always a partition of type κ compatible with $(A_1, \ldots, A_m)$ if there is one compatible with the system $(A'_1, \ldots, A'_m)$. Then, for each type κ, the number of such partitions for $(A_1, \ldots, A_m)$ is at least as large as the number of such partitions for $(A'_1, \ldots, A'_m)$.*

We can show by network flow arguments that the condition "for each type κ, the number of type κ partitions compatible with $(A_1, \ldots, A_m)$ is at least as large as the number of such partitions compatible with $(A'_1, \ldots, A'_m)$" is equivalent to the condition:

For each $I \subset \{1, \ldots, m\}$, we have

$$|\bigcup_{i \in I} A_i| \geq |\bigcup_{i \in I} A'_i|.$$

From this, the relation to our matching problem is apparent.

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$(A_1, \dots, A_n)$ and $A' := (A'_1, \dots, A'_n)$ are subsets of some n-element set W, $U := \{1, \dots, n\}$. Assume further that

$$|\bigcup_{i\in I} A_i| \geq |\bigcup_{i\in I} A'_i| \quad \text{for all } I \subset U.$$

Then the number of transversals for A is at least as large as the number of transversals for A'. What if equality holds in the inequality above for all subsets I of U? If the conjecture is true, both systems must have the same number of transversals. In fact, we will prove in a moment that (in case of equality for *all* I) the systems A and A' differ only by a permutation of the ground set W:

Each of the systems A and A' generates a Boolean algebra with atoms $A_I := \bigcap_{i\in I} A_i \cap \bigcap_{i\notin I} A_i^c$ and $A'_I := \bigcap_{i\in I} A'_i \cap \bigcap_{i\notin I} A_i'^c$, $I \subset U$, respectively. It suffices to show that the cardinalities $f(I) := |A_I|$ and $f'(I) := |A'_I|$ are equal for all I. Now, consider the functions g and g' from 2^U to the nonnegative integers defined by $g(I) := \sum_{J\subset I} f(J)$ and similarly for g'. Denoting by I^c the complement of I in U, we have for all $I \subset U$:

$$g(I) = \sum_{J\subset I} f(J) = n - \sum_{J : J\cap I^c \neq \emptyset} f(J) = \bigcup_{i\in I^c} A_i = \bigcup_{i\in I^c} A'_i = g'(I).$$

The functions g and g' are thus equal. But f and f' arise from g and g' by Moebius inversion on the power set lattice and, hence, are equal too.

The conjecture would follow from a much more general conjecture by Aharoni and Keich (cf. [1]). See also [3] for a similar conjecture) which we are going to state now:

Suppose we are given two nonnegative, real $(m \times n)$-matrices $A = (a_{i,j})$ and $A' = (a'_{i,j})$. If $X := (X_1, \dots, X_m)$ is an ordered *set partition* of $\{1, \dots, n\}$, we call the ordered *number partition* $\kappa := (k_1, \dots, k_m)$ of n with $k_i := |X_i|$ the *type* of X. Denote by $a(X)$ the expression

$$a(X) := \prod_{i=1}^{m} \prod_{j\in X_i} a_{i,j}$$

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6 Ervin Győri

Does the book function of graphs jump at every rational number?

Following the terminology introduced by Sheehan, a collection of k triangles in a graph G is called a book of k pages if they share a common edge.

Let $b(c)$ be the function defined in the interval $[1/2, 1]$ as the maximum b such that every graph G of (sufficiently large) n vertices and with minimum degree $cn + o(n)$ contains a book of bn pages.

The function b obviously is monotone increasing and it is easy to see that it is continuous from the left hand side, since deleting edges changes the sizes of the books sitting on the remaining edges as "spine" edges, and if an edge e is the spine of an exceptionally thick book then instead of this edge, delete other edges incident to the endvertices of e (e.g., the other edges of this book).

For a rational number $1/2 \leq p/q < 1$, let us consider the product representation

$$\frac{p}{q} = \frac{k_1 - 1}{k_1} \cdot \frac{k_2 - 1}{k_2} \cdot \ldots \cdot \frac{k_r - 1}{k_r}$$

such that $k_i > (k_i - 1)^2$ for $i = 1, 2, ..., r-1$. (It is easy to see that every rational number $1/2 \leq p/q < 1$ has a unique product representation of this type.) Then let

$$f(\frac{p}{q}) = \frac{k_1 - 2}{k_1} \cdot \frac{k_2 - 2}{k_2} \cdot \ldots \cdot \frac{k_r - 2}{k_r}$$

and extend this function f in the irrational numbers so that it should be continous at every irrational number. (It can be done in a unique way, as well.) I conjecture that $b = f$. The conjecture is a consequence of Turán's classical theorem if $r = 1$ in the product representation of p/q and is proved in [1] if $r = 2$. The proof of the case $r = 2$ is very complicated, about more then 20 pages. This proof might be generalized for $r = 3$ in 30-40 pages, but the general case probably needs new ideas.

The following purely graph theoretical conjecture is in fact equivalent to Conjecture 1:

Conjecture 2 *Assume that a (proper) coloring of a cycle of length* $n \geq 2m - 1$ *with* m *colours is given. Add to the cycle all the edges between points of the same colour. Then the resulting graph has a second Hamiltonian cycle.*

Meanwhile it was proved by Carsten Thomassen (cf. [4] and [5]) that Conjecture 2 is true if all colour classes have at least three vertices. He also provided a counterexample to the general conjecture, namely, Fleischner's graph in [2], Figure 6. This graph has 28 vertices and may be coloured by 13 colours such that 11 colours appear twice and two colours are used thrice. It yields in a canonical way an example of a TSP as described above with $m = 13$ and 28 cities where the only optimal tour contains $28 = 2m + 2$ edges of positive length.

Of course, the question of a good general upper bound for the number of positive edges in an optimum tour remains.

16.2 A matching inequality

The next problem can be stated as follows:

Conjecture 3 *Suppose that* $G = (V, E)$ *and* $G' = (V, E')$ *are bipartite graphs on the same vertex set* V *and with the same 2-colouring* $V = U \cup W$ *where both colour classes* U *and* W *contain* n *elements. Assume further that for all* $A \subset U$ *the number of neighbours in* G *is at least as large as in* G'*:*

$$|\Gamma_G(A)| \geq |\Gamma_{G'}(A)| \quad \textit{for all } A \subset U.$$

Then the number of perfect matchings in G *is at least as large as the number of perfect matchings in* G'*.*

Note that the case where G' is 1-regular is just the famous "marriage theorem" by P. Hall. Instead of bipartite graphs and matchings we could as well talk about set systems and their transversals and get the following reformulation of our conjecture: Suppose that $A :=$

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16 Eberhard Triesch

Conjectures on the uniqueness of Hamiltonian cycles and the number of perfect matchings

16.1 Cement mixers and Hamiltonian Cycles

Consider the following vehicle routing problem: A cement mixer visits a certain collection of customers. It starts at some depot where it is filled and after each visit it has to be filled again at one of m depots. When all customers were visited, it returns to its starting point. What is the length of a shortest tour? In [6] it is shown that the problem is essentially equivalent to the following version of a Traveling Salesman Problem:

Suppose that n cities $c_1, \ldots, c_n$ are given as well as distances $d(c_i, c_j) \geq 0$ between the cities with the additional property that the collection of cities can be partitioned into m disjoint subsets $M_1, \ldots, M_m$ such that for all $1 \leq i \leq m$ the following condition holds:

$$d(c_j, c_k) = 0 \quad \text{for all} \quad c_j, c_k \in M_i$$

This problem is NP-complete if m is part of the input, but it is polynomially solvable if m is considered to be constant (as is suggested by the application). It is easy to see (cf. [6]) that an optimum tour may contain at most $m(m-1)$ "edges" c_i, c_j of positive "weight" $d(c_i, c_j)$. From this it follows that the TSP is solvable in $O(n^{2m(m-1)+1})$ time. However, as was remarked in [6], the estimate $m(m-1)$ for the number of edges of positive weight in an optimum tour seems to be too pessimistic. Since we could only construct examples with at most $2m-2$ edges, we came up with the following conjecture which I also offered at the IV Prague Workshop:

Conjecture 1 *In a TSP of the type described above, an optimum tour can have at most $2m-2$ edges of positive weight.*

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7 Andreas Huck

Cycle–double–covers in graphs without Petersen–graph–minor

The well–known cycle–double–cover–conjecture states that each bridgeless graph G has a k–cycle–double–cover $(C_1, ..., C_k)$ for some integer k, i.e. each C_i is an Eulerian subgraph of G (not necessarily connected) and each edge of G is contained in C_i for exactly two indices $i \leq k$. [1] proved the following.

Theorem 1: Each bridgeless graph not containing the Petersen–graph P_{10} as a minor has a k–cycle–double–cover for some integer k.

The proof of Theorem 1 in [1] does not lead to a polynomial algorithm constructing k–cycle–double–covers. Moreover, a stronger version of the cycle–double–cover–conjecture states that each bridgeless graph even contains a 5–cycle–double–cover. So it is near to try to strengthen Theorem 1 as follows.

Conjecture 1: Each bridgeless graph without P_{10} as a minor has a 5–cycle–double–cover which can be found in polynomial time.

Recently, [2] proved the following.

Theorem 2: Each cubic graph G without P_{10} as a minor has girth at most 5, i.e. there exists a circuit of length at most 5 in G.

Using this result, Conjecture 1 can be verified for cubic graphs. Similarly, Conjecture 1 without any restrictions can be deduced from the following conjecture which is an analogy of Theorem 2 for arbitrary graphs.

Conjecture 2: Each graph of minimal degree at least 3 without P_{10} as a minor has girth at most 6.

In this conjecture, we cannot replace 6 by 5 since there exist so–called apex–graphs of girth 6 and of minimal degree at least 3. Apex–graphs can be made planar by deleting a certain vertex and do not contain P_{10} as a minor.

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8 Martin Klazar

Towards an extremal theory of colored trees?

Suppose T is a finite tree and $f : V(T) \to \mathbf{N}$ is a coloring of its vertices. We remind that the coloring is *proper* if no edge is monochromatic. We introduce two conditions forbidding certain color patterns in (T, f).

The coloring (T, f) is *tripod-free* if there are no four distinct vertices u_1, u_2, u_3, and v such that $f(u_1) = f(u_2) = f(u_3) \neq f(v)$ and each two of the u_i-v paths intersect only in v. In other words, there is no subgraph of T which is a subdivision of the star with four vertices and in which f colors the three endvertices by the same color, different to the color of the central vertex.

Note that such an edge exists if G has another Hamiltonian cycle. In [5] it was proved that every Hamiltonian graph of minimum degree at least 3 has another Hamiltonian cycle. This means that a smallest counterexample to Conjecture 2 must be nonbipartite. But then it would be also a counterexample to Conjecture 1. In other words, Conjecture 1 implies Conjecture 2.

An old Conjecture of Sheehan [3] says that every 4-regular Hamiltonian graph has another Hamiltonian cycle. At the IV Prague Workshop, Eberhard Triesch made the following conjecture:

If a cycle C with n vertices is colored in m colors, $2m < n + 1$, and we add an edge between any two vertices of the same color, then the resulting graph has a Hamiltonian cycle distinct from C.

In [6] the conjecture is verified if all color classes have at least 3 vertices. The graph of Fleischner [2], Figure 6, shows that the conjecture is false even if all color classes have at least two vertices.

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Jackson also proved that 32/27 is best possible in the sense that there are chromatic polynomials with roots arbitrary close to 32/27. I extended this last statement as follows.

Theorem 2. [4] The roots of the chromatic polynomials consist of 0, 1, and a dense subset of the interval $(32/27, \infty)$.

It follows from Jackson's result that for every 2-*connected* graph G with an odd number of vertices we have

$$P(G, 32/37) < 0.$$

So if, in addition, G is bipartite, then

$$P(G, 2) > 0$$

and hence $P(G, x)$ has a root in $(32/27, 1)$. However, Jackson made the following conjecture:

Conjecture 1. [1]. If G is 3-connected and nonbipartite, then $P(G, x)$ has no root in $(1, 2)$.

I suggested the following conjecture which, if true, would establish a new connection between colorings and Hamiltonian cycles.

Conjecture 2. [5]. If G has a Hamiltonian cycle, then $P(G, x)$ has no root in $(1, 2)$. More precisely, $(-1)^n P(G, x)$ is positive in $(1, 2)$, where $n = |V(G)|$.

It is easy to reduce Conjecture 2 to the 3-connected case, see [5]. If e is an edge in a graph G, then G/e denotes the graph obtained by contracting e. It is well known and easy to see that

$$P(G, x) = P(G - e, x) - P(G/e, x).$$

Hence Conjecture 2 would follow from an affirmative answer to the following

Question. Suppose G is a 3-connected Hamiltonian graph. Does G contain an edge e such that both $G - e$ and G/e are Hamiltonian?

Similarly, (T, f) is *abba-free* if there are no four distinct vertices u_i, $i = 1, \ldots, 4$, such that u_2 and u_3 lie on the u_1-u_4 path and $f(u_1) = f(u_4) \neq f(u_2) = f(u_3)$. In other words, if a path in T has the endpoints colored by the same color c then c is the only color which may be repeated on the path.

Problem 1. Let (T, f) be a proper coloring that uses at most n colors and that is tripod-free and *abba*-free. Find, in terms of n, the maximum number of vertices of T. Or at least prove (or disprove) that then $|V(T)| = O(n)$.

It is instructive to look at the problem when T is restricted to a path. Then the tripod condition is void and the maximum size of $V(T)$ is not difficult to determine — it equals to $3n-2$. We leave this to the sympathetic reader as an exercise. In the tree case we cannot obtain even the $O(n)$ upper bound.

Now we formulate a more general conjecture. Let j be a positive integer. We say that (T, f) is $a^j b^j a^j b^j$*-free* if there is no path $(v(1), v(2), \ldots, v(k))$ in T such that, for some $4j$ indices $1 = i_1 < i_2 < \cdots < i_{4j} = k$ and two distinct colors a and b,

$$\begin{aligned} a &= f(v(i_1)) = f(v(i_2)) = \cdots = f(v(i_j)) = \\ &= f(v(i_{2j+1})) = f(v(i_{2j+2})) = \cdots = f(v(i_{3j})) \end{aligned}$$

and

$$\begin{aligned} b &= f(v(i_{j+1})) = f(v(i_{j+2})) = \cdots = f(v(i_{2j})) = \\ &= f(v(i_{3j+1})) = f(v(i_{3j+2})) = \cdots = f(v(i_{4j})). \end{aligned}$$

Problem 2. Let (T, f) be a proper coloring that uses at most n colors and that is tripod-free and $a^j b^j a^j b^j$-free, where j is a fixed positive integer. Prove (or disprove) that then $|V(T)| = O(n)$.

In the restricted path case the $O(n)$ bound is valid, see [1]. More information about the path case can be found in [2].

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9 Martin Kochol

Snarks are cyclically 4-edge-connected cubic graphs with girth at least 5 and chromatic index 4. Note that a graph is *cyclically* k*-edge connected* if deleting fewer than k edges does not disconnect the graph into components so that at least two of them have circuits. *Girth* of a graph is the length of its shortest circuit.

Snarks play an important role in graph theory since it is known that the smallest counterexamples to the *Cycle Double Cover Conjecture* and the 5-*Flow Conjecture* of Tutte must ne snarks.

We have constructed cyclically 5-edge connected snarks with arbitrary large girths (see [5, 6]), thereby obtaining a counterexample to a conjecture of Jaeger and Swart [2], where was conjectured that every snark has girth at most 6. There are constructions of cyclically 6-edge-connected snarks with girths 6 (see [1, 4, 7]). Therefore we can set the following problem:

Problem Construct cyclically 6-edge-connected snarks with arbitrary large girths.

Note that there is another known conjecture of Jaeger and Swart [3], where is conjectured that there are no snarks with cyclical edge-connectivity greater that 7.

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15 Carsten Thomassen

Roots of chromatic polynomials and another Hamiltonian cycle

In [1] Bill Jackson proved the following intriguing result.

Theorem 1. If G is a graph and $P(G,x)$ is its chromatic polynomial, then G has no root $\leq 32/27$ except 0 and 1.

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10 Alexandr V. Kostochka[0] and Michael Stiebitz

On the number of edges in colour-critical graphs

Recall that a graph G is called k-critical if the chromatic number of G is k and each proper subgraph of G is $(k-1)$-colourable. Let $\mathcal{C}(k,n)$ denote the set of k-critical graphs on n vertices. Clearly,

$$\min\{d_G(x) \mid x \in V(G),\, G \in \mathcal{C}(k,n)\} = k-1.$$

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Define *degree excess* $\epsilon(G)$ for $G \in \mathcal{C}(k,n)$ as follows:

$$\epsilon(G) := \sum_{x \in V(G)} (d_G(x) - (k-1)) = 2|E(G)| - (k-1)n.$$

Below, k is an integer not less than 4. By Brooks' theorem, $\epsilon(G) > 0$ for each $G \in \mathcal{C}(k,n)$ if $n > k$. G. A. Dirac [1] proved

Theorem 1 *If $n > k$ then $\epsilon(G) \geq k-3$ for every $G \in \mathcal{C}(k,n)$.*

Shorter proofs were given by Kronk and Mitchem [3] and Weinstein [5]. The bound is attained by all graphs in the family $\mathcal{D}(k)$ that is defined as follows. Each member D of $\mathcal{D}(k)$ has $2k-1$ vertices partitioned into three subsets: V', V'' and $\{v_1, v_2\}$ where $|V'| = k-2$, $|V''| = k-1$, the subgraphs of D induced by $V' \cup \{v_1\}$, $V' \cup \{v_2\}$, and V'' are complete graphs, each vertex in V'' is adjacent to exactly one of v_1 and v_2, each of v_1 and v_2 is adjacent to some vertex in V'', and there are no other edges. (See Fig. 1)

In 1974, Dirac [2] extended Theorem 1 as follows.

Theorem 2 *Let $n > k$. Then for every $G \in \mathcal{C}(k,n) \setminus \mathcal{D}(k)$,*

$$\epsilon(G) \geq \begin{cases} 2, & \text{if } k = 4; \\ k-1, & \text{if } k > 4. \end{cases}$$

Mitchem [4] gave a shorter proof of this theorem.

Define the family $\mathcal{F}(k)$ as follows. Each member F of $\mathcal{F}(k)$ has a vertex v such that $V(F) - v = W \cup U$, where $|W| = |U| = k-1$, $F(W) \simeq F(U) \simeq K_{k-1}$ and each vertex in $W_2 = \{w \in W \mid (v,w) \notin E(F)\}$ is adjacent to each vertex in $U_2 = \{u \in U \mid (v,u) \notin E(F)\}$. There are no other edges in $F - v$. (An example is shown on Fig. 2.)

If $W_2 \neq \emptyset$, $U_2 \neq \emptyset$, and $|W_2| + |U_2| \leq k-1$, then F is k-critical. Clearly, $\mathcal{D}(k) = \{F \in \mathcal{F}(k) \mid |W_2| = 1 \text{ or } |U_2| = 1\}$. For each s, $k-3 \leq s \leq 2(k-3)-1$ with $s \equiv k-3 \pmod 2$, there is a graph $F \in \mathcal{F}(k)$ with $\epsilon(F) = s$. For example, we can take a graph $F \in \mathcal{F}(k)$ with $|W_2| = 2$ and $|U_2| = (s+5-k)/2$.

Very recently, we proved the following refinement of Dirac's theorem.

Theorem 3 *Let $n > k$. Then for every $G \in \mathcal{C}(k,n) \setminus \mathcal{F}(k)$,*

$$\epsilon(G) \geq 2(k-3).$$

with the property that the disjunction of any $\leq r$ $(r \geq 2)$ codewords are distinct (UD_r codes). This led them to studying the binary codes with the property that none of the codewords is covered by the disjunction of $\leq r$ others (*Superimposed codes, ZFD$_r$ codes;* P. Erdős, P. Frankl and Z. Füredi called the correspondig set system r-cover-free in [7]).

Since that many results have been proved about the maximum size of these families. Various authors studied these problems basically from three different points of view, and these three lines of investigations were almost independent of each other. This is why many results were found first in information theory ([1], [4], [5], [14], [15], [16], [17]), were later rediscovered in combinatorics ([2], [6], [7], [10], [18], [19], [20]), or in group testing ([12], [13]), and vice versa.

About the maximum size if these families is known the following.

Theorem.

$$2^{(c_1/r^2)n} \leq T^\star(r,n) \leq T(r,n) \leq T'(r,n) \leq 2^{(c_2 \log r/r^2)n},$$

where c_1 and c_2 are two (absolute) constants.

The lower bound was proved in [1, 4, 5, 7, 13], while the upper bound was proved by the author [19]. Hwang and Sós [13] asked what is the gap between those three functions. The author [19] proved that $T(r,n)$ and $T'(r,n)$ are essentially the same. Therefore, we would like to ask a similar question to one asked by Hwang and Sós.

Problem. *Does the inequality*

$$T(r,n) \leq \sqrt[c]{T^\star(r,n)}$$

for some (absolute) constant c holds?

If this is true, then it would mean that r-cover-free families are essentially big distance codes, and hence it is hard to find this maximum, since the asymptotic behaviour of the big distance codes is one of the major (more then 40 years!) open problems in coding theory. On the other hand, if this is not true, it would yeald to a $\omega(1/r^2)$ (in the exponent!) lower bound for the maximum size of the r-cover-free families. Note that to give such a lower bound is a longstanding open problem as well.

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14 Miklós Ruszinkó

Are r-cover-free families essentially big distance codes?

Let S be an n-element set and 2^S is the set of all subsets of S. We call $\mathcal{F}' \subset 2^S$ r-distinct, if

$$\bigcup_{i=1}^{k} A_i \neq \bigcup_{j=1}^{\ell} B_i$$

for any

$$\{A_1, A_2, \ldots, A_k\} \neq \{B_1, B_2, \ldots, B_\ell\},$$

$1 \leq k, \ell \leq r$; $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_\ell \in \mathcal{F}'$. $\mathcal{F} \subset 2^S$ is r-cover-free, if

$$A_0 \not\subseteq A_1 \cup A_2 \cup \ldots \cup A_r$$

holds for all distinct $A_0, A_1, \ldots, A_r \in \mathcal{F}$. $\mathcal{F}^\star \subset 2^S$ is $< r$ part intersecting, if

$$|A_i \cap A_j| < \frac{1}{r} \min\{|A_i|, |A_j|\}$$

for any distinct $A_i, A_j \in \mathcal{F}^\star$ holds. We denote by $T'(r,n)$, $T(r,n)$, $T^\star(r,n)$ the maximum cardinality of the corresponding set systems. We will provide bounds on these functions for r fixed and n tending to infinity.

The notion of the r-cover-free families was introduced by Kautz and Singleton in 1964 [17]. They initiated investigating binary codes

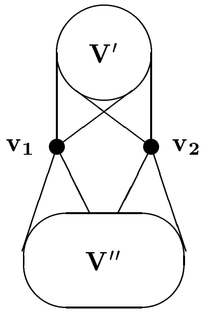

Fig. 1: A graph in $\mathcal{D}_k$.

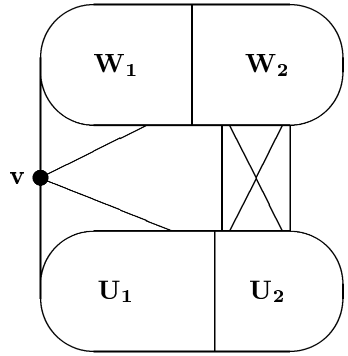

Fig. 2: A graph in $\mathcal{F}_k$.

This bound is attained on some graphs in $\mathcal{C}(k,n)$ for each $n \in \{k+2, 2k-2, 2k, 3k-1\}$. Our conjecture is the following.

Conjecture. *If* $n > k$ *and* $n \notin \{k+2, 2k-2, 2k-1, 2k, 3k-1\}$, *then* $\epsilon(G) \geq 3(k-3)$ *for each* $G \in \mathcal{C}(k,n)$.

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11 Jan Kratochvíl, Jaroslav Nešetřil, and Moshe Rosenfeld

Matchability of Hadamard Matrices

11.1 Spanning graph designs

The 1-skeleton of the icosahedron and the great icosahedron are isomorphic graphs. A search for explanation of this well known fact has led us to the beautiful theory of equiangular line systems and related graphs developed by J.J.Seidel [5, 6, 7]. This theory centers around the notion of graph switching. For more details we refer the reader to [3]. Motivated by this we consider in [3] graph equations of the

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If $\Phi(e) \neq 0$ for every $e \in E$, Φ is called a nowhere-zero $\mathbb{Z}_k$-flow.

A *cycle double cover* of a graph $G = (V, E)$ defined on E is a family of cycles of G such that each element of E appears in exactly two cycles of the family. A cycle double cover is said $k-$*colorable*, if it is a k-cycle cover. It means that we can color the circuits of the cover in such way that each element of E appears in two circuits of different colors. For more details concerning the cycle double cover of graphs see [2, §5].

The following conjectures are well known [1, 6, 3, 4, 5]:

Conjecture 1 *Every bridgeless graph G has a nowhere-zero* $\mathbb{Z}_5$*-flow.*

Conjecture 2 *Every bridgeless graph G has a* $5-$*colorable cycle double cover.*

Conjecture 3 *Every bridgeless graph G has a cycle cover with length at most* $|E| + |V| - 1$.

We propose the following weaker conjectures:

Conjecture 4 *If a bridgeless graph G has a nowhere-zero* $\mathbb{Z}_5$*-flow then it has a cycle cover with length at most* $|E| + |V| - 1$.

Conjecture 5 *If a graph G has a nowhere-zero* $\mathbb{Z}_5$*-flow then it has a* $5-$*colorable cycle double cover.*

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Remark that this process may generate graphs that have no representation by intersection of arcs : Let G_0 be a graph with at least three vertices and one edge. Then, there exists a graph G, obtained from G_0 by applying a sequence of operations II, such that G is not an intersection graph of arcs, as 15 operations are sufficient to obtain an induced subgraph which is a total subdivision of $K_{3,3}$.

Problem 4 *Let G be a planar graph satisfying*

$$\forall H \subseteq G, \quad |E(H)| \leq 2|V(H)|.$$

Is G a contact graph of geodesics of the sphere?

Problem 5 *If G is an intersection graph of intervals of pseudolines, is G an intersection graph of segments?*

Remember that if G is a contact graph of intervals of pseudolines, then G is a contact graph of segments. But there are intersection graphs of arcs that are not intersection graphs of segments (e.g. $K_{n,n}$ without a perfect matching, for $n > 5$).

13 Andre Raspaud

Cycle Covers and Nowhere-Zero Flows

We denote the *vertex-* and *edge-sets* of a graph G by E and V, respectively. A *circuit* is a connected 2-regular subgraph and a *cycle* is a union of edge-disjoint circuits. A *cycle cover* of a graph G is a family of cycles of G such that each edge of G is contained in at least one cycle of the family; a cycle cover is called a *k-cycle cover* if it is formed by at most k cycles. The *length* of a cycle cover $\mathcal{C}$ is the sum of the lengths (number of edges) of the cycles in $\mathcal{C}$.

Let $G = (V, E)$ be a simple graph, for a given orientation of it edges, if $S \subseteq V$ we denote by $\omega^+(S)$ (resp. $\omega^-(S)$) the set of the edges with the inital (resp. terminal) vertex in S and the other one in $V - S$. A $\mathbb{Z}_k$*-flow* Φ ($k \geq 1$) is a mapping from E to $\mathbb{Z}_k$ such that

$$\forall v \in V \sum_{e \in \omega^+(\{v\})} \Phi(e) = \sum_{e \in \omega^-(\{v\})} \Phi(e)$$

---

following types:

$$K_{2n} = 2G(2n, n-1) + nK_2 \quad (1)$$

$$(n-1)K_{2n} = (2n-1)G(2n, n-1) \quad (2)$$

($G(n,m)$ denotes an m-regular graph on n vertices). While the solutions of equation (1) can be completely characterized (in terms of switching equivalence), the problem arising from the related equation (2) appears to be much harder. We believe that this particular graph equation (which can be also interpreted as a graph design in the sense of [2]) is interesting even in rather special cases of the simplest $G(2n, n-1)$ graphs. For $G(2n, n-1) = 2K_n$, we showed in [3] that (2) is equivalent to the existence of Hadamard matrices of order $2n$. For $G(2n, n-1) = K_{n,n} - nK_2$ (sometimes called the Hiraguchi graph), the situation is more involved. In this case we relate the problem of solvability of (2) to a new concept: *matchability* of Hadamard matrices (defined in the next section).

11.2 Matchability of column balanced matrices

Definition A ± 1 matrix is called *column balanced* if every column contains the same number of symbols $+1$ and -1. Let A be a column balanced $2n \times m$ matrix and let $d_{i,j}$ denote the Hamming distance of the i-th and j-th row of A. Counting the sum of these distances columnwise, we see that

$$\sum_{i,j \in \{1,2,\ldots,2n\}} d_{i,j} = m \cdot n^2.$$

We say that A is *matchable* if it is possible to equalize the row distances by removing a perfect matching in each column, i.e., if there exist m sets $M_k \subset \binom{\{1,2,\ldots,2n\}}{2}$, each containing n disjoint pairs, such that

(i) $\{i,j\} \in M_k$ implies $A_{i,k} \cdot A_{j,k} = -1$ and

(ii) $d_{i,j} - |\{k : \{i,j\} \in M_k\}| = \dfrac{m(n-1)}{2n-1}$.

(The matchings are understood as a device to reduce distances of rows of A. If the i-th and j-th row differ in the k-th coordinate, then the occurrence of $\{i,j\}$ in M_k reduces the distance d_{ij} by one. The total reduction is then mn and the sum of the reduced distances is $mn^2 - mn$. Hence $\frac{m(n-1)}{2n-1} = \frac{mn^2-mn}{\binom{2n}{2}}$ is the average reduced row distance in A. Obviously, the number of columns of a matchable matrix is divisible by $2n-1$ and a necessary condition for a $2n \times m$ matrix to be matchable is that the Hamming distance of any two rows is $\geq \frac{m(n-1)}{2n-1}$.) The role of these notions is captured by the following observation:

Observation 11.1 [3] *The graph $K_{n,n} - nK_2$ solves equation (2) if and only if there exists a column balanced matchable matrix of type* $2n \times (2n-1)$.

Example. The 4×3 matrices

$$A = \begin{pmatrix} +1 & +1 & +1 \\ +1 & -1 & -1 \\ -1 & +1 & -1 \\ -1 & -1 & +1 \end{pmatrix} \qquad B = \begin{pmatrix} +1 & +1 & +1 \\ +1 & -1 & +1 \\ -1 & +1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

are column balanced. The matchings are depicted in Figure 1. Therefore $K_{2,2} - 2K_2 = 2K_2$ solves (2) for $n = 2$.

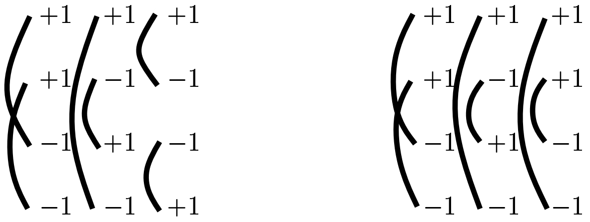

Figure 1: Column balanced matchable matrices of type 4×3.

Incidentally, in the transposed form the pattern of Figure 1 goes back to Ch. Morgenstern (Fig. 2):

(II) *for an edge $\{v_1, v_2\}$ and a vertex u distinct from v_1 and v_2, add a vertex x, add three edges $\{x, v_1\}$, $\{x, v_2\}$ and $\{x, u\}$, and delete the edge $\{v_1, v_2\}$.*

C_3: *G is a CCS graph and $|E(G)| = 2|V(G)| - 3$.*

1 ⇒ 2: this is a theorem of Tay and Whiteley (*Structural Topology*, **11**, p. 21–69).
2 ⇒ 3: for any of the two operations, it is easy to construct a representation of the new CCS graph either by simply adding a segment, or by adding a segment and adjusting one existing segment. The condition on the number of the edges is also satisfied.
3 ⇒ 1: if we look at the segments representing the vertices of H, there are at least three end points which are not on any other segment.

12.2 Problems

Although any contact graph of segments in $\mathbb{R}^3$ has obviously a representation as a contact graph of crossing segments in the plane, some graphs which have a representation by contacts of crossing segments are not intersection graphs of segments in $\mathbb{R}^3$. Consider for instance a K_7 and subdivide all the edges of an induced K_5. In any segment-intersection representation all the segments are coplanar leading to a contradiction.

Problem 1 *Let G be a graph, satisfying*

$$\forall H \subseteq G, \quad (|V(H)| > 2), \qquad |E(H)| \leq 2|V(H)| - 4.$$

Is G a contact graph of segments in $\mathbb{R}^3$?

Problem 2 *Which are the graphs that may be obtained from the triangular prism by a sequence of operation II (as described in Theorem 1)?*

Problem 3 *Does any sequence of operations II performed on the triangular prism correspond to a sequence of plane truncations of the triangular prism?*

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12 Tsuyoshi Matsumoto

A few taxiplanie problems

12.1 A Problem already solved

A graph G is representable by *contacts of crossing segments* and is called a *CCS graph* if the vertices $v_1, \ldots, v_n$ of the graph can be represented by possibly crossing segments $s(v_1), \ldots, s(v_n)$ in such a way that two vertices v_i and v_j are adjacent if, and only if, one end point of one of the two segments $s(v_i)$ and $s(v_j)$ belongs to the interior of the other segment.

If the segments have no crossing, G is a contact graph of segments.

Theorem 1 *Let G be a connected graph. The following conditions are equivalent:*

C_1: *G satisfies the following conditions:*

- $|E(G)| = 2|V(G)| - 3$.
- *For any subgraph H of G with at least two vertices,* $|E(H)| \leq 2|V(H)| - 3$.

C_2: *G can be obtained from K_2 by applying the following two operations:*

(I) for two distinct vertices v_1 and v_2, add a vertex x and two edges $\{x, v_1\}$ and $\{x, v_2\}$;

Figure 2: Fisches Nachtgesang.

11.3 Matchability of Hadamard and Conference matrices

Certain classical combinatorial structures yield matchable column balanced matrices of type $2n \times (2n-1)$. The truncation of a Hadamard matrix in standard form is obtained by deleting the first all-1 column. A Hadamard matrix is *matchable* if its truncation is matchable. (Note that since all row distances are the same in Hadamard matrices, such a matrix is matchable iff every pair of rows is matched exactly once). We conjecture that every Hadamard matrix is matchable, and we support our conjecture by showing that some well known standard constructions of Hadamard matrices give matchable matrices.

Let q be a prime power. We denote by $\mathbf{1} = (1, 1, \ldots, 1)$ the all-one vector (of appropriate length, which will be clear from context), and by I_n the identity matrix of size $n \times n$. Let $\chi(x)$ denote the quadratic residue character over $GF(q)$ (i.e., $\chi(0) = 0$, $\chi(x) = +1$ iff $x = y^2$ for some $y \in GF(q) - \{0\}$ and $\chi(x) = -1$ otherwise). We define the following matrices over $GF(q)$:

$$R = (\alpha - \beta)_{\alpha, \beta \in GF(q)},$$

$$Q = (\chi(\alpha - \beta))_{\alpha, \beta \in GF(q)},$$

$$H'_{q+1} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1}^T & Q \end{pmatrix} + I_{q+1}.$$

As shown in [1], H'_{q+1} is a Hadamard matrix of order $q+1$ if

$q \equiv 3 \bmod 4$. The equivalent Hadamard matrix in standard form is

$$H_{q+1} = \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{1}^T & -Q - I_q \end{pmatrix}.$$

Theorem 1 [3] *If $q \equiv 3 \bmod 4$ then the Hadamard matrix H_{q+1} is matchable.*

When $q \equiv 1 \bmod 4$ the Hadamard matrix H derived from Q is more complex. We first form the matrix:

$$S_{q+1} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1}^T & Q \end{pmatrix}.$$

As shown in [1], this is a symmetric conference matrix of order $q+1$. Let A be the $(q+1) \times q$ matrix obtained from $S_{q+1} - I_{q+1}$ by deleting the first column. We know that A is a column balanced matrix of type $(q+1) \times q$ and the Hamming distance between the first row and any other row is $\frac{q+1}{2}$. It is easy to see that $dist_H(A^\alpha, A^\beta) = \frac{q+1}{2} + \chi(\alpha - \beta)$. One can see (cf. [3]) the following:

Theorem 2 *The matrix A is matchable.*

Paley's construction then gives a Hadamard matrix of order $2q+2$. The matchability of these matrices consitutes a considerably harder problem which was not yet completely solved. One of the partial results is the following:

Theorem 3 *If $q \equiv 5 \bmod 8$ then the Hadamard matrix H_{2q+2} is matchable.*

The following lemma parallels the well known result stating that tensor product of Hadamard matrics is again a Hadamard matrix.

Lemma 11.2 *If a Hadamard matrix H' is matchable then $H_2 * H'$ is also matchable.*

Corollary 11.3 *For every $n = 2^k$, $k \geq 1$, a Hadamard matrix of order n exists and $G = K_{n,n} - nK_2$ solves (2).*

Corollary 11.4 *For every $n = (q+1) \cdot 2^k$, such that $k \geq 0$ and $q \equiv 3 \bmod 4$, a Hadamard matrix of order n exists and $G = K_{n,n} - nK_2$ solves (2).*

11.4 Open problems

The most appealing is the general conjecture:

Problem 1. Is every Hadamard matrix matchable? Or at least, is it true that if a Hadamard matrix of order $4m$ exists then there also exists a matchable Hadamard matrix of the same order?

While this conjecture seems quite challenging, the following questions suggested by W. Wallis and R. Craigen may be more tractable:

Problem 2. If H is a matchable Hadamard matrix in standard form, is the transpose of H matchable as well?

Problem 3. If we exchange the first column of a matchable Hadamard matrix H with another column and multiply the appropriate rows by -1 to transform the resulting matrix to standard form, is the new Hadamard matrix matchable?

Though Hadamard matrices became our primary interest, we conclude with possibly an equally important problem:

Problem 4. Find necessary and sufficient conditions for a column balanced matrix to be matchable. What is the computational complexity of deciding whether a given column balanced matrix is matchable?

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