On Infimum of Optimal Objective Function Values in Interval Linear Programming

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Abstract: The paper considers linear programming problem whose coefficients are uncertain. We shall suppose that all input data may vary independently and simultaneously in given intervals. Theoretical background for calculating the exact lower and upper bounds of optimal values of an objective function for such a problem is given with an emphasis to the first one. The theory leads to algorithms for computing these bounds.

Key words: linear programming problem, inexact data, interval coefficients.

0 Introduction

It occurs in practical linear programming (abbr. LP) problems that the input data are uncertain. There have been several approaches how to solve such problems with a different description of the data, e.g. parametric , stochastic, or by using fuzzy sets. We shall consider independent and simultaneous variations of all coefficients within given intervals. Problems of this type will be called interval linear programming (abbr. ILP) problems. In this paper we turn our attention to calculating the exact lower and upper bounds of all optimal values. The problem was solved by using interval arithmetic, e.g. by Beeck [1], Jansson [2], Krawczyk [4], or without this tool by Rohn [12] and [13]. We shall use the latter approach. A structure of the paper is as follows. Some notation and calculation of the exact upper bound is mentioned in Section 1 briefly. A theoretical background for calculating the exact lower bound is given in Section 2. The last part contains corresponding algorithms and some comments regarding computational aspects.

1 Notations and Calculating Upper Bound

Because of interval description of an uncertain data we shall consider an interval m by n matrix $[A]$ and an interval $m-$vector $[b]$ given by

$$[A] = \{A \in \mathrm{R}^{mn} : \underline{A} \leq A \leq \overline{A}\} \tag{1}$$

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$$[b] = \{b \in \mathrm{R}^m : \underline{b} \le b \le \overline{b}\} \tag{2}$$

with known bounds $\underline{A}$, $\overline{A}$, $\underline{b}$ and $\overline{b}$, and with an assumption $\underline{b} < \overline{b}$ throughout the paper. The center matrix of $[A]$ is given by $A_c = (\overline{A} + \underline{A})/2$ and the radius matrix by $\Delta = (\overline{A} - \underline{A})/2$. The center vector b_c and the radius vector δ of interval $[b]$ are defined in an analogous way.

By *interval linear programming* problem we mean a family of LP problems

$$S(A,b): \qquad \sup\{c^T x : Ax = b, x \ge 0\}, \tag{3}$$

where $c \in \mathrm{R}^n$ is a given vector, $A \in [A]$ and $b \in [b]$.

An LP problem $S(A,b)$ with fixed $A \in [A]$ and $b \in [b]$ is called a *subproblem of an ILP problem*

For each subproblem $S(A,b)$ we shall denote by $X(A,b)$ the set of its feasible solutions, i.e.

$$X(A,b) = \{x \in \mathrm{R}^n : Ax = b, x \ge 0\}. \tag{4}$$

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There are two special kinds of subproblems which play an important role. They are introduced in a following way. Let T denote the set of m-vectors t with $|t_i| \le 1, i = 1, ..., m$ and let $T_t = \mathrm{diag}\{t_1, ..., t_m\}$ be the diagonal matrix with vector $t = (t_i)$ on the diagonal. A subproblem $S(A_t, b_t)$, where

$$A_t = A_c + T_t\Delta \quad \text{and} \quad b_t = b_c - T_t\delta, \quad t \in T \tag{5}$$

will be denoted by S_t and called a *t-subproblem* of an ILP problem.
As a special case, let H denote the set of m-vectors h with $|h_i| = 1, i = 1, ..., m$. A subproblem S_h, $h \in H$ will be called an *extremal subproblem.* Its i-th constraint has the form

$$(\overline{A}x)_i = \underline{b}_i \quad \text{if} \quad h_i = 1 \quad \text{and} \quad (\underline{A}x)_i = \overline{b}_i \quad \text{if} \quad h_i = -1.$$

These equations will be called the opposite extremal ones. The sets of feasible solutions $X(A_t, b_t)$ and $X(A_h, b_h)$ will be denoted by using abbreviations X_t and X_h, respectively.

Let us denote by X the set of all feasible solutions for a given ILP problem, i.e.

$$X = \cup\{X(A,b) : A \in [A], b \in [b]\}. \tag{6}$$

A relation between sets X and X_t is described by an assertion which was proved in [5].

Lemma 1 *If x is an element of X, then there is a unique $t \in T$ such that $x \in X_t$.*

Let $f(A,b)$ be the optimal value of a subproblem $S(A,b)$:

$$f(A,b) = \sup\{c^T x : x \in X(A,b)\}. \tag{7}$$

The function f defined in this way is called the *solution function* of an ILP problem. Thus, the exact upper and lower bounds of optimal values of all subproblems for an ILP problem are given by the values

$$\overline{f} = \sup\{f(A,b) : A \in [A], b \in [b]\} \tag{8}$$

$$\underline{f} = \inf\{f(A,b) : A \in [A], b \in [b]\}. \tag{9}$$

Remark Nonnegativity of feasible solutions imply that the values $\overline{f}$ and $\underline{f}$ are reached for $c = \overline{c}$ and $c = \underline{c}$, resp. Therefore we need not consider an interval vector

$$[c] = \{c \in \mathrm{R}^n : \underline{c} \leq c \leq \overline{c}\}$$

for coefficients of an objective function. □

As the values $\overline{f}$ and $\underline{f}$ are reached in some vertices of X (except of unboundedness of an objective function), we give a characterization of these vertices.

Theorem 1 *If x is a vertex of some X_h, $h \in H$, then x is a vertex of X. If x is a vertex of the set X, then either x is a vertex of some X_h, $h \in H$, or x does not belong to any X_h . In this case, x is a vertex of some X_t, $t \in T - H$.*

Proof was given in [5] by using a characterization of vertices of a convex polytope proved in Theorem 13 from [8]. □

The set X will be called *regular*, if each its vertex is a vertex of some $X_h, h \in H$. More details about regularity and other properties of the set X are given in [5].

It is well-known - see e.g. [9] - that

$$X = \{x \in \mathrm{R}^n : \underline{A}x \leq \overline{b}, \overline{A}x \geq \underline{b}, x \geq 0\}. \tag{10}$$

This expression of the set X implies that the exact upper bound of optimal values can be found quite easily. The problem of calculating this value is equivalent to the following problem (see [1]):

$$\sup\{c^T x : x \in X\}. \tag{11}$$

Because of this result the exact upper bound of optimal values can be computed in a polynomial time. Nevertheless, the constraint matrix of the LP problem (11) has the number of rows doubled in a comparison to the original interval matrix $[A]$. Therefore, algorithms working with tableau of a smaller size may be of some interest. They are described e.g. in [6], or by Rohn in [12] and [13].

2 Calculating the Exact Lower Bound

Opposite to the result (11), a calcualtion of the exact lower bound is much more difficult. It follows immediately from the following theorem proved by Rohn in [14].

Theorem 2 *Computing the value $\underline{f}$ within accuracy 1/2 is NP-hard for rational data $A, \underline{b}, \overline{b}, c$ and for a finite value of $\underline{f}$.*

For a given m- vector $y = (y_i)$ we shall introduce an $m \times n$ matrix A_y by defining its i-th row and m-vector b_y by defining its i-th component in the following way:

If $y_i \geq 0$, then $(A_y)_i = \overline{A}_i$ and $(b_y)_i = \underline{b}_i$,
if $y_i < 0$, then $(A_y)_i = \underline{A}_i$ and $(b_y)_i = \overline{b}_i$.

The symbols A_t, b_t, A_h, b_h will keep the meaning which is defined in the previous section.

Lemma 2 *For each $y \in R^m, A \in [A], b \in [b]$ we have*

$$\begin{aligned} b^T y &\geq b_y^T y & (12) \\ A^T y &\leq A_y^T y. & (13) \end{aligned}$$

Lemma 3 *For each $y \in R^m, h \in H$ the following are equivalent*

$$\begin{aligned} b_h^T y &= b_y^T y & (14) \\ y_i h_i &\geq 0, \quad i = 1, ..., m & (15) \\ A_h^T y &= A_y^T y. & (16) \end{aligned}$$

Proofs of Lemma 2 and Lemma 3 follow immediately from the above definitions of A_y and b_y. □

Lemma 4 *Let y be a feasible solution of an LP problem $D(A, b)$ which is dual to $S(A, b)$. Let $h \in H$ satisfy condition (15). Then y is a feasible solution to the problem D_h which is dual to S_h.*

Proof. By the assumption, $A^T y \geq c$. Using Lemma 2 and Lemma 3, we have

$$A_h^T y = A_y^T y \geq A^T y \geq c,$$

which was to prove. □

Lemma 5 *Let $S(A, b)$ have a finite optimal value with a dual optimal solution y_0. Let an extremal subproblem (S_h) have a finite optimal value, where $h \in H$ satisfies condition (15). Then $f(A, b) \geq f(A_h, b_h)$ holds.*

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Proof. Due to the Duality theorem it holds $f(A,b) = b^T y_0$ and the value $f(A_h, b_h)$ is equal to the optimal value of the problem D_h. Previous Lemma ensures that y is a feasible solution to the problem D_h. By using Lemma 2 and Lemma 3 we have

$$f(A,b) = b^T y_0 = b_{y_0}^T y_0 = b_h^T y_0 = \min\{b_h^T y : A_h^T y \geq c\} = f(A_h, b_h). \tag{17}$$

□

Given ILP problem is called *strongly solvable*, if each subproblem $S(A,b)$ has a finite optimum. Necessary and sufficient conditions for strong solvability were given by Rohn in [11].

Theorem 3 *Let all extremal subproblems of an ILP problem are feasible. Then the ILP problem is strongly solvable if and only if the problem (11) has a finite optimum.*

The following theorem proves that the value $\underline{f}$ can be reached by solving a finite number of subproblems under the assumption of strong solvability.

Theorem 4 *Let an ILP problem be strongly solvable. Then there is $h^* \in H$ such that $\underline{f} = f(A_{h^*}, b_{h^*}) = \min\{f(A_h, b_h) : h \in H\}$.*

Proof. By the assumption, all subproblems have a finite optimal objective function value. Then for each $(A,b) \in [A,b]$ there is an $h \in H$ satisfying condition (15) and due to the previous Lemma we have $f(A,b) \geq f(A_h, b_h)$. The set H has a finite number of elements, which completes the proof. □

Note. The assumption of strong solvability in Theorem 4 cannot be omitted. It is proved by the following example. The value $\underline{f}$ is reached at points of the set X which do not belong to a set of feasible solutions of any extremal subproblem.
□

Example 1

Let us consider an ILP problem given by

$$\underline{\mathbf{A}} = \begin{pmatrix} 4 & 2 & 2 \\ 4 & 6 & 8 \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} 8 & 4 & 6 \\ 6 & 10 & 12 \end{pmatrix}, \quad \underline{\mathbf{b}} = \begin{pmatrix} 20 \\ 36 \end{pmatrix}, \quad \overline{\mathbf{b}} = \begin{pmatrix} 28 \\ 44 \end{pmatrix}$$

with an objective function $c^T x = x_1 + x_2 + 3x_3$.

The extremal subproblem with $h = (-1,-1)$ has the optimal value $41/3$, the extremal subproblem with $h = (1,1)$ has the optimal value $44/5$. Next two extremal subproblems have no feasible solution. There is, however, a subproblem given by

$$\mathbf{A} = \begin{pmatrix} 4 & 4 & 6 \\ 6 & 6 & 10 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 24 \\ 36 \end{pmatrix},$$

whose optimal value 6 is less than optimal values of extremal subproblems. □

Theorem 4 leads to an idea to find the exact lower bound by solving all 2^m extremal subproblems. It can be done by using ILP algorithm which is described in [6] because it enables an easy passage from an extremal subproblem to another one. It will be discussed in more details later. As this procedure means to solve an exponential number of extremal subproblems we shall describe another possibility by formulating an optimality criterion for reaching the exact lower bound $\underline{f}$. We shall use a close connection between the value $\underline{f}$ and an optimal value of the problem

$$\inf\{b_y^T y : A_y^T y \geq c\}. \tag{18}$$

It is not a usual LP problem as coefficients in the objective function and constraints matrix of the problem (18) depend on signs of vector y. To see a structure of the problem (18) let us rewrite it in an equivalent form

$$\min\{\inf\{b_h^T y : A_h^T y \geq c, y_i h_i \geq 0, \quad i = 1, ..., m\} : h \in H\}. \tag{19}$$

For a fixed value $h \in H$, the condition $y_i h_i \geq 0, \quad i = 1, ..., m$ describes exactly one orthant in the space R^m.

Lemma 6 *Let y be a feasible solution of the problem*

$$D(A, b) : \qquad \inf\{b^T y : A^T y \geq c\}. \tag{20}$$

Then y is a feasible solution to the problem (18) and it holds

$$\inf\{b^T y : A^T y \geq c\} \geq \inf\{b_y^T y : A_y^T y \geq c\}. \tag{21}$$

Proof. Due to (13) y is a feasible solution of the problem (18). The second part follows immediately from (12). □

Lemma 7 *Let there be $(A, b) \in [A, b]$ such that the problem $D(A, b)$ is feasible and unbounded. Then the problem (18) is unbounded, too.*

Proof. Let us denote by $Y(A, b)$ the set of feasible solutions of the problem $D(A, b)$. Due to the assumption there are $y \in Y(A, b)$ and a vector $u \in R^m$ such that $y(r) = y + ru$ belongs to the set $Y(A, b)$ for each nonnegative real value r with

$$\lim_{r \to \infty} b^T y(r) = -\infty.$$

By Lemma 2, $y(r)$ is a feasible solution of the problem (18) for each nonnegative $r \in R$ and (12) implies that

$$\lim_{r \to \infty} b_{y(r)}^T y(r) = -\infty.$$

□

(i) We omit the added row and we pass to another extremal subproblem, if condition (15) is not satisfied for more indices.

(ii) We start the algorithm with another initial extremal subproblem.

(iii) We continue carefully by the algorithm INF1 with counting number of iterations to check a possible cycling.

There are also positive experiences with using the assertion of Theorem 4, i.e. solving all 2^m extremal subproblems for calculating the exact lower bound. For a small value m it can be done by using the INF1 algorithm. Due to Theorem 7 it enables to pass quickly from an optimal solution of a current extremal subproblem to a feasible solution (very often even optimal) of another one. Thus, the algorithm INF2 was constructed with a systematic passage through all extremal subproblems and with Warning, if the set X is not regular. Moreover, there are some properties of an ILP problem that can be characterized in terms of extremal subproblems, namely strong solvability, basic stability, boundedness, nonregularity -see [5]. Therefore, the use of the algorithm INF2 can give all of these informations with a very small additional work. To finish a computation in a reasonable time, however, the value m should be less than 20.

Algorithm INF2

0. Choose a $h \in H$ and set $locmin := Largevalue$.
1. Using the simplex method compute the optimal solution x^h of the subproblem S_h and a dual optimal solution y.
2. If $y_i h_i \geq 0, i = 1, ..., m$ and $c^T x^h < locmin$ then set $locmin := c^T x^h$.
3. If all extremal subproblems are solved then set $\underline{f} := locmin$ and STOP.
3. Using steps 3.1 through 3.5 pass to the new extremal subproblem, whose constraints are systematically changed exactly in one row. STOP with a Warning if the index of pivot row $r = m + 1$.
4. Go to step 1.

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If an ILP problem is not basis stable, then condition (15) is sufficient at least for local minimizer of the solution function in a following sense. By a *stable region with a basis* B we shall mean a set

$\Re_B = \{(A, b) \in [A, b] : S(A, b)$ has a unique nondegenerate optimal solution with the basis $B\}$.

Theorem 13 *Let x^h with a basic index set B be a nondegenerate optimal solution of an extremal subproblem S_h with a dual optimal solution y satisfying (15). Then*

$$c^T x^h = \inf\{f(A, b) : (A, b) \in \Re_B\}. \tag{34}$$

Proof. By Theorem 1, x^h is a vertex of the set X. Let B and N be index sets of basic and nonbasic variables of the point x^h, resp. Let us introduce the set

$$X_N = \{x \in X : x_j = 0, j \in N\}.$$

This set is a convex subset of the set X. Because of Theorem 8 we have: Each vertex neighbouring to the vertex x^h in the set $X - X_h$ is an element of the set X_N. Let x_0 be such a vertex. Then $c^T x_0 \geq c^T x^h$ holds due to Theorem 9. As the objective function is linear and X_N is a convex polytope, it holds for each $x \in X_N$:

$$c^T x \geq c^T x^h. \tag{35}$$

Let us consider the set

$$X_B^{opt} = \{x : x \text{ is an optimal solution of } S(A, b) \text{ for some } (A, b) \in \Re_B\}.$$

Definition of the stable region $\Re_B$ implies that $X_B^{opt} \subset X_N$. Hence (35) holds for each $x \in X_B^{opt}$, which proves (34). □

Because of the previous Theorem we have the following conclusion if the algorithm INF1 terminates in step 2. The current element (A_h, b_h) gives the exact lower bound of the solution function at a stable region with a basis B.

There is still a problem of a termination of the algorithm INF1. An implementation of the algorithm INF1 for computer was done. Numerical experiences with an implementation on computer confirmed that the algorithm works successfully, except of an ILP problem with a nonregular set X. This case, however, can be recognized. It is a situation when a pivot calculated in step 3.4 does not lie in the added row by performing step 3 of the algorithm - see Theorem 7. If such a case appears, a label *Warning* must be signaled as cycling may occur. Therefore, the algorithm INF1 terminates after a finite number of steps with Warning, or with finding the lower bound at least on a stable region with a current basis B. If the case Warning appears, there are several possibilities how to proceed:

Theorem 5 *Let an ILP problem be strongly solvable. Then we have*
$$b_{y^*}^T y^* = \underline{f},$$
where y^* is an optimal solution of the problem (18).

Proof. An optimal solution y^* to the problem (18) exists because of the assumption of strong solvability. Moreover, the Duality theorem and Lemma 6 imply that $b_{y^*}^T y^* \leq \underline{f}$ holds.

On the other hand, strong solvability assumption, definition of $\underline{f}$ and the Duality theorem imply

$$\underline{f} = \inf\{\inf\{b^T y : A^T y \geq c\} : (A, b) \in [A, b]\}. \tag{22}$$

Then we have

$$\underline{f} \leq \min\{\inf\{b_h^T y : A_h^T y \geq c\} : h \in H\}. \tag{23}$$

Therefore,

$$\underline{f} \leq \min\{\inf\{b_h^T y : A_h^T y \geq c, y_i h_i \geq 0, \quad i = 1, ..., m\} : h \in H\}. \tag{24}$$

It means that

$$\underline{f} \leq b_{y^*}^T y^*, \tag{25}$$

which completes the proof. □

The following assertion gives a necessary condition for calculating the exact lower bound $\underline{f}$.

Theorem 6 *Let an ILP problem be strongly solvable. Let x be an optimal solution of an extremal subproblem S_h with $c^T x = \underline{f}$. Then a dual optimal solution $y^* = (y_i^*)$ satisfies condition (15).*

Proof. An existence of an element x is ensured by Theorem 4. Then, there is a dual optimal solution y^*, which is a feasible solution to the problem (18) due to Lemma 6 and we have

$$b_{y^*}^T y^* \geq \inf\{b_y^T y : A_y^T y \geq c\} = \underline{f} = c^T x = b_h^T y^* \geq b_{y^*}^T y^*,$$

where the previous Theorem, the assumption, the duality theorem and Lemma 2 were used. Thus, condition (14) holds, which is equivalent to (15). □

Theorem 5 brings an idea of the algorithm for calculating lower bound $\underline{f}$: to solve a sequence of extremal subproblems until the optimality criterion (15) is satisfied.

Algorithm INF1

0. Choose a $h \in H$.

1. Using simplex method compute an optimal solution x^h of the subproblem S_h and a dual optimal solution y.

2. If $y_i h_i \geq 0, i = 1, ..., m$, then STOP.

3. Specify the index s with $y_s h_s = \min\{y_i h_i : y_i h_i < 0\}$ and pass to the new subproblem, whose constraints are changed in exactly the s-th row.

4. Go to step 1.

Let us denote by x^k the optimal solution to an extremal subproblem S_h computed in step 1. Let x^{k+1} be the basic feasible solution corresponding to the simplex tableau after a passage to a new subproblem by step 3.

Theorem 7 *The feasible solution x^{k+1} is a vertex of the set X. If the pivot row is the added one, then x^k and x^{k+1} have the same basic variables and x^{k+1} is a vertex of the set X_h of the new subproblem with the opposite extremal s-th equation. Otherwise, x^{k+1} is a vertex of the set X_t with $t \in (-1, 1)$. In both cases we have $c^T x^{k+1} < c^T x^k$, if the previous subproblem is nondegenerate.*

Proof. The first part was proved in [6] by using characterization of vertices of the set X. As regards the objective function value, the assertion follows immediately from the well-known formulas for simplex transformation. □

A passage to a new subproblem in step 3 can be done efficiently by adding the opposite s-th extremal equation to the constraints. The details are described in [7]. For a reader's convenience we mention this procedure here briefly. We introduce a new artificial basic variable x_{m+n+s} in the added eqaution and execute an usual step of the simplex method by determining a pivot from the $(n+s)$-th column. If the pivot lies in the added row, then variable x_{m+n+s} leaves the basis in the following step. We must change the value h_s to the opposite one as we have moved into a new point satisfying the added extremal equation. We can omit the added row after a transformation of the tableau because the artificial variable x_{n+s} becomes basic. If the pivot does not lie in the added row, we have passed into the point, which satisfies neither of two opposite extremal s-th equations - see Theorem 7. It means that the set$X$ is not regular and the algorithm terminates with a Warning. The above procedure can be described as follows.

STEP 3:

3.1 If $h_s = -1$, then add new $(m+1)$-th equation $(\overline{A}x)_s + x_{m+n+s} = \underline{b}_s$ to the tableau, set $C := 1$ and go to step 3.3.

3.2 Add the new $(m+1)$-th equation $(\underline{A}x)_s + x_{m+n+s} = \overline{b}_s$ to the tableau, and set $C := -1$.

3.3 Substitute the expressions of the basic variables from the first m rows into the added row, where x_{m+n+s} becomes a basic variable.

3.4 Perform a usual transformation of the simplex tableau with $n+s$ as the index of pivot column.

3.5 If the index of pivot row $r = m+1$, then set $t_s := -t_s$ and drop out the added row, otherwise STOP with Warning.

Remark. The substitution in step 3.3 can be performed very easily by a repeated simplex transformation of the added equation, where the unit coefficients in basic columns are successively taken as the pivots. Due to Theorem

If an ILP problem is basis stable then due to (32) and (10) we can express the set X^{opt} in a following way:

$$X^{opt} = \{x \in \mathrm{R}^n : (\underline{A})_B x_B \le \overline{b}, (\overline{A})_B x_B \ge \underline{b}, x_B \ge 0, x_N = 0\}.$$

Hence, X^{opt} is a convex polytope and (33) is the LP problem. □

Under a basic stability assumption we can prove that condition (15) is sufficient for reaching the exact lower bound.

Theorem 11 *Let an ILP problem be basis stable. Let x^h be an optimal solution of an extremal subproblem S_h, whose dual optimal solution y satisfies condition (15). Then $c^T x^h = \underline{f}$ holds.*

Proof. By the assumption, x^h is a nondegenerate vertex of the sets X and X^{opt}. Let $x_0 \ne x^h$ be any vertex of the set X^{opt}. Then x_0 is a vertex of some extremal subproblem $S_{h'}$. If x_0 is a neighbouring vertex to x^h then (30) holds due to Theorem 9. To summarize we have: X^{opt} is a convex polytope, the objective function is linear and the relation (30) holds for any vertex of the set X^{opt} which is a neighbouring vertex to the x^h. Therefore, x^h is an optimal solution to the problem (33). □

Assumption of basic stability ensures the finiteness of the algorithm INF1, too.

Theorem 12 *Let an ILP problem be basis stable. Then there is $h \in H$ such that an optimal solution y^h of the problem D_h satisfies the optimality criterion (15) and $\underline{f} = b_h^T y^h$ holds. This value is reached by the INF1 algorithm in a finite number of steps.*

Proof. A basic stability assumption ensures that the ILP problem is strongly solvable. The first part of the assertion holds due to Theorem 6 and the Duality Theorem. Let x^k be an optimal solution to some extremal subproblem S_h. Suppose that its dual optimal y does not satisfy (15) for index s. The basic stability assumption ensures that the set X is regular (see [5]). Then Theorem 7 implies that the execution of step 3 leads to a basic feasible solution x^{k+1} of a new extremal subproblem with the opposite extremal s-th equation. Moreover, x^{k+1} has the same basic variables as the previous point x^k and $c^T x^{k+1} < c^T x^k$. The asumption of basic stability implies that x^{k+1} is the optimal solution of the new extremal subproblem. As an ILP problem has a finite number of extremal subproblems, the INF1 algorithm terminates in a finite number of steps with the value $\underline{f}$ in accordance to Theorem 4. □

Remark Under basic stability assumption, after a passage through step 3 of the algorithm INF1 we have an *optimal solution* of the next subproblem with a smaller optimal value. It follows from theorem 7. It is an important feature of the algorithm that this result is reached without verifying assumption of basic stability. □

Proof. A satisfaction of the condition (15) means that for each index $i \in \{1, 2, ..., m\}$ the exactly one of the following possibilities holds:

$$\text{Either} \quad y_i > 0, h_i = 1, \quad \text{or} \quad y_i < 0, h_i = -1, \quad \text{or} \quad y_i = 0. \qquad (31)$$

Because of Theorem 1 there is a unique $t \in T$ such that x_0 is a vertex of the set X_t. Due to Theorem 8 it holds: $t \neq h$, vectors t, h differ in exactly one component (let us denote it by s) and for any index i it holds: if the i-th component of x^h is equal zero then the i-th component of x_0 is equal zero, too. Hence, it is possible to pass from the point x^h to the point x_0 by following the steps 3.1 through 3.5 of the algorithm INF1, see also Theorem 7. Then the inequality (30) is satisfied because of (31), expression for determining the pivot row in step 3 of the algorithm INF1 and expression for objective function value

$$c^T x_0 = c^T x^h - (y_s b_r)/a_{r,n+s}.$$

□

Let us turn our attention to the set of optimal solutions of all subproblems for a given ILP problem denoted by X^{opt}, i.e. to the set

$$X^{opt} = \{x^* : x^* \text{ is an optimal solution of } S(A, b) \text{ for some } A \in [A], b \in [b]\}.$$

Suppose that a given ILP problem is basic stable. Let the set of basic indices and the set of nonbasic indices be denoted by B and N, resp. By X_B we denote the set of nonnegeative solutions of a system of interval linear equations $[A_B]x_B = [b]$. Following proposition was formulated by Beeck in [1].

Theorem 10 *Let an ILP problem be basis stable. Then it holds*

$$X^{opt} = \{x \in \mathrm{R}^n : x_B \in X_B, x_N = 0\}. \qquad (32)$$

Proof. Suppose x_0 satisfies conditions describing the set in (32). Then $x_0 \in X$ and hence x_0 is a feasible solution of some subproblem $S(A, b)$. The assumption of basic stability ensures a strong solvability of a given ILP problem and moreover, x_0 is an optimal solution of the mentioned subproblem. Thus x_0 belongs to the set X^{opt}. Conversely, let x_0 be an element of the set X^{opt}. The assumption of basic stability implies that $(x_0)_i = 0$ for $i \in N$. As we have $x_0 \in X^{opt} \subset X$, there are $A \in [A], b \in [b]$ such that $A_B(x_0)_B = b$. Hence, $(x_0)_B$ belongs to X_B and (32) holds. □

Remark Definition of the set X^{opt} implies that

$$\underline{f} = \inf\{c^T x : x \in X^{opt}\}. \qquad (33)$$

7 a label Warning in step 3.5 signals that the set X is not regular. Such a case will be discussed later. □

There is a problem what is possible to say if the algorithm INF1 terminates in step 2, because condition (15) is not sufficient for reaching the value of infimum $\underline{f}$. It is proved by the following example, where a given ILP problem is even strongly solvable.

Example 2

Let us consider an ILP problem given by

$$\underline{\mathbf{A}} = \begin{pmatrix} 4 & 2 & 2 \\ 4 & 6 & 8 \end{pmatrix}, \quad \overline{\mathbf{A}} = \begin{pmatrix} 8 & 2.5 & 6 \\ 4.5 & 10 & 12 \end{pmatrix}, \quad \underline{\mathbf{b}} = \begin{pmatrix} 20 \\ 36 \end{pmatrix}, \quad \overline{\mathbf{b}} = \begin{pmatrix} 28 \\ 44 \end{pmatrix}$$

with an objective function $c^T x = x_1 + x_2 + 3x_3$.

All extremal subproblems have a feasible solution. The extremal subproblem with $h = (1, -1)$ has an optimal solution at the point $(0, 13/2, 5/8)$ with the optimal value 67/8 and the dual optimal solution $y = (5/8, -3/32)$. It means that condition (15) is satisfied. The extremal subproblem with $h = (-1, 1)$ has an optimal solution at the point $(88/13, 0, 6/13)$ with the dual optimal solution $y = (-1/26, 10, 39)$. The condition (15) is satisfied, too, but the optimal function value 106/13 is less than for the previous extremal subproblem. □

To receive a sufficient condition, let us notice that optimal solutions of two extremal subproblems in Example 2 have not the same basic variables. Therefore we shall turn our attention to a case when such a situation does not happen. It leads to the following definiton. An ILP problem is called *basis stable with basis* B, if each subproblem $S(A, b), A \in [A], b \in [b]$ has a unique nondegenerate optimal basic solution with the basis B.

We shall verify that optimality criterion (15) is sufficient for reaching the exact lower bound $\underline{f}$ under the assumption of basic stability. Moreover, it ensures that algorithm INF1 terminates after a finite number of steps with calculating the value $\underline{f}$. First, we shall prove three propositions regarding

- Description of neighbouring vertices of the set X of all feasible solutions.
- Objective function values in neighbouring vertices of the set X.
- Description of the set of optimal solutions for all subproblems, if a given ILP problem is basis stable. □

Vertices $x^1, x^2 \in X$ are called neighbouring if they are vertices of the same edge of the set X. By Theorem 13 in [8] it happens if and only if any element of the set

$$Y = \{x \in \mathrm{R}^n : x = \lambda x^1 + (1 - \lambda)x^2, \lambda \in (0, 1)\} \qquad (26)$$

satisfies at least $n - 1$ inequalities of the system

$$\underline{A}x \le \overline{b}, \overline{A}x \ge \underline{b}, x \ge 0 \qquad (27)$$

as equations and the rank of a coefficients matrix of such a system is equal $n-1$.

Theorem 8 *Let x^1, x^2 be vertices of the set X, let x^1 be a nondegenerate. Then x^1 and x^2 are neighbouring vertices of the set X if and only if either*

(i) x^1 and x^2 are neighbouring vertices of the same extremal set X_h,

or

(ii) there are $h \in H, t \in T, t \neq h$ such that x^1 is a vertex of the set X_h, x^2 is a vertex of the set X_t, the vectors h, t differ in exactly one component and for any index i it holds: if the i-th component of x^1 is equal zero then the i-th component of x^2 is equal zero, too.

Proof. Let x^1 be a nondegenerate vertex of the set X. It is easy to verify that there is the $h \in H$ such that x^1 is a nondegenerate vertex of the set X_h. To prove a part "if" we should verify that (under the assumptions) a segment connecting points x^1 and x^2 is an edge of the set X. First, let (ii) hold. Assume that

$$h_i = t_i = 1, \quad i = 1, 2, ..., m-1, \quad h_m = 1, t_m \neq 1.$$

It means that

$$(\underline{A}x^1)_i = (\underline{A}x^2)_i = \overline{b}_i, \quad i = 1, 2, ..., m-1.$$

Choosing any $x \in Y$ we have

$$(\underline{A}x)_i = \underline{A}(\lambda x^1 + (1-\lambda)x^2)_i = \lambda \overline{b}_i + (1-\lambda)\overline{b}_i = \overline{b}_i, \quad i = 1, 2, ..., m-1. \quad (28)$$

As x^1 is a nondegenerate it has exactly $n-m$ zero components. Due to the assumption (ii), these components are equal zero for x^2, too. Therefore, each element of the segment Y has the same property. Together with (28) it means that each point of the segment Y satisfy $n-1$ inequalities of the system (27) as equations. Let us denote by K the matrix of coefficints of these $n-1$ equations. Then K contains a unit submatrix of the size $n-m$ created by the coefficients of equations $x_i = 0$. Moreover, K contains a submatrix A' of coefficients of the first $m-1$ equations of the system $\underline{A}x = \overline{b}$. Matrix $\underline{A}$ has a rank m because of nondegeneracy of the point x^1. Thus, matrix A' has a rank $m-1$ which imply that matrix K has a rank $n-1$. It proves that x^1 and x^2 are neighbouring vertices of the set X.

If (i) holds then the part "if" can be proved in a similar way.

To prove the part "only if" let x^1 and x^2 be neighbouring vertices of the set X and let x^1 be a nondegenerate. Then there is $h \in H$ such that x^1 is a nondegenerate vertex of the set X_h and we can suppose that $h_i = 1, \quad i = 1, 2, ..., m$. Moreover, there is a unique $t \in T$ such that x^2 is a vertex of the set X_t . If $t = h$ holds, then both x^1 and x^2 are solutions of the system $\underline{A}x = \overline{b}$. As the point x^1 has exactly $n-m$ zero components, x^2 has at least $n-m-1$ of these components equal zero, too. If not then points of the segment Y do not satisfy at least $n-1$ inequalities of the system (27) as equations, a contradiction to the fact that x^1 and x^2 are neighbouring vertices of the set X. Therefore, any point of the segment Y satisfies all m equations of the system $\underline{A}x = \overline{b}$ and moreover, it has at least $n-m-1$ zero components. In an analogous way to the previous paragraph we can verify a satisfaction of the condition regarding a rank. We can conclude that x^1 and x^2 are neighbouring vertices of the set X_h, as the segment Y is the edge of this set.

It remains to prove the assertion in a case when $t \neq h$. To prove it by a contradiction let us suppose that the point x^1 has exactly the first m components positive (in accordance to the nondegeneracy assumption) and the last $n-m$ components equal zero, while $(x^2)_q > 0$ for some $q > m$. Then we have for each $x \in Y$:

$$\begin{aligned} x_i &= \lambda(x^1)_i + (1-\lambda)(x^2)_i > 0 \quad i = 1, 2, ..., m, \\ x_q &= (1-\lambda)(x^2)_q > 0. \end{aligned} \quad (29)$$

Therefore any point $x \in Y$ has at most $n-m-1$ zero components. As $t \neq h$ holds we can suppose that $h_m = 1, t_m \neq 1$. It means that

$$(\underline{A}x^1)_m = \overline{b}_m, \quad (\underline{A}x^2)_m < \overline{b}_m.$$

Then it holds for each $x \in Y$.

$$(\underline{A}x)_m = \underline{A}(\lambda x^1 + (1-\lambda)x^2)_m = \lambda\overline{b}_m + (1-\lambda)(\underline{A}x)_m < \lambda\overline{b}_m + (1-\lambda)\overline{b}_m = \overline{b}_m.$$

In a similar way we can verify that $(\overline{A}x)_m > \underline{b}_m$ because the assumption $\underline{b} < \overline{b}$ implies that $(\overline{A}x^1)_m > \underline{b}_m$. Hence, each point of the segment Y satisfies at most $m-1$ extremal equations. Together with (29) it means that each point of the segment Y satisfies at most $n-2$ inequalities of the system (27) as equations. It implies that the segment Y is not an edge of the set X, a contradiction to the assumption that x^1 and x^2 are neighbouring vertices of the set X. Thus, the i-th component of x^2 is equal zero if the i-th component of x^1 is equal zero. Analogously to the above part we can verify that vectors h, t cannot differ in more than one components. Namely, if they differ at least in two components then any point of the segment Y satisfies at most $m-2$ extremal equations and it has at most $n-m$ zero components. It leads to the contradiction with the fact that the segment Y is an edge of the set X. □

Using the assertion of the last Theorem we shall prove a relation between objective function values in neighbouring vertices of the set X.

Theorem 9 *Let x^h be a nondegenerate optimal solution of an extremal subproblem X_h whose dual optimal solution satisfies optimality criterion (15). Let x_0 be a neighbouring vertex to the point x^h in the set $X - X_h$. Then we have*

$$c^T x_0 \geq c^T x^h. \quad (30)$$