The QuaSEFE Problem

Patrizio Angelini, Henry Förster, Michael Hoffmann, Michael Kaufmann, Stephen Kobourov, Giuseppe Liotta, Maurizio Patrignani

27th International Symposium on Graph Drawing and Network Visualization 2019
The QuaSEFE Problem

QuaSEFE
The QuaSEFE Problem
The QuaSEFE Problem

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

QuaSEFE

Simultaneous (Graph) Embedding with Fixed Edges

Input: Set of planar graphs with shared vertex set
The QuaSEFE Problem

**Input:** Set of planar graphs with shared vertex set

**Output:** Planar drawings for all graphs such that

![Diagram of QuaSEFE problem]
The QuaSEFE Problem

**Input:** Set of planar graphs with shared vertex set

**Output:** Planar drawings for all graphs such that vertices have the same position in all drawings (simultaneous drawings)
The QuaSEFE Problem

**Input:** Set of planar graphs with shared vertex set

**Output:** Planar drawings for all graphs such that

- vertices have the same position in all drawings (simultaneous drawings)
- edges have the same representation in all drawings (fixed edges)
The QuaSEFE Problem

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

Quasiplanarity

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

- Quasiplanarity
- Quasiplanar Embedding: No triple of edges crosses pairwise

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

- Quasiplanarity
- Quasiplanar Embedding: No triple of edges crosses pairwise

 forbidden
The QuaSEFE Problem

- Quasiplanar Embedding: No triple of edges crosses pairwise

- Quasiplanarity

Simultaneous (Graph) Embedding with Fixed Edges

- forbidden
- allowed (no triple)
The QuaSEFE Problem

Quasiplanarity

Simultaneous (Graph) Embedding with Fixed Edges

- Quasiplanar Embedding: No triple of edges crosses pairwise

- Thickness two drawings (i.e. two-edge colorable drawings) are quasiplanar
The QuaSEFE Problem

Quasiplanarity

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

- Quasiplanarity

- QuaSEFE Problem:

Simultaneous (Graph) Embedding with Fixed Edges
The QuaSEFE Problem

Quasiplanarity

Simultaneous (Graph) Embedding with Fixed Edges

QuaSEFE Problem:

\textbf{Input:} Set of quasiplanar graphs with shared vertex set
The QuaSEFE Problem

QuaSEFE Problem:
- **Input:** Set of *quasiplanar* graphs with shared vertex set
- **Output:** Simultaneous *quasiplanar* drawings for all graphs with fixed edges

Quasiplanarity
Related Work

- always positive instances for SEFE
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area)  [Brass et al. ’06]
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area) \[\text{[Brass et al. '06]}\]
  - a planar graph and a tree \[\text{[Frati '06]}\]
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area) [Brass et al. ’06]
  - a planar graph and a tree [Frati ’06]
- counterexamples for SEFE
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area)
  - a planar graph and a tree
- counterexamples for SEFE
  - three paths

[Brass et al. ’06]
[Frati ’06]
[Brass et al. ’06]
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area) [Brass et al. ’06]
  - a planar graph and a tree [Frati ’06]
- counterexamples for SEFE
  - three paths [Brass et al. ’06]
  - two outerplanar graphs [Frati ’06]
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area)  \[\text{[Brass et al. '06]}\]
  - a planar graph and a tree  \[\text{[Frati '06]}\]
- counterexamples for SEFE
  - three paths  \[\text{[Brass et al. '06]}\]
  - two outerplanar graphs  \[\text{[Frati '06]}\]
- SEFE testable in $O(n^2)$ time for two biconnected planar graphs with connected intersection  \[\text{[Bläsius & Rutter '16]}\]
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area) [Brass et al. ’06]
  - a planar graph and a tree [Frati ’06]
- counterexamples for SEFE
  - three paths [Brass et al. ’06]
  - two outerplanar graphs [Frati ’06]
- SEFE testable in $O(n^2)$ time for two biconnected planar graphs with connected intersection [Bläsius & Rutter ’16]
- Variants
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area) [Brass et al. ’06]
  - a planar graph and a tree [Frati ’06]
- counterexamples for SEFE
  - three paths [Brass et al. ’06]
  - two outerplanar graphs [Frati ’06]
- SEFE testable in $O(n^2)$ time for two biconnected planar graphs with connected intersection [Bläsius & Rutter ’16]
- Variants
  - no fixed mapping between vertices [Brass et al. ’06]
Related Work

- always positive instances for SEFE
  - two caterpillars (in polynomial area) [Brass et al. ‘06]
  - a planar graph and a tree [Frati ‘06]
- counterexamples for SEFE
  - three paths [Brass et al. ‘06]
  - two outerplanar graphs [Frati ‘06]
- SEFE testable in $O(n^2)$ time for two biconnected planar graphs with connected intersection [Bläsius & Rutter ‘16]
- Variants
  - no fixed mapping between vertices [Brass et al. ‘06]
  - geometric simultaneous embedding (GSE) [Angelini et al. ’11, Di Giacomo et al. ’15]
Related Work - SEFE and Beyond Planarity

- quasiplanar GSE
Related Work - SEFE and Beyond Planarity

- quasiplanar GSE
  - a tree and a cycle [Didimo et al. ’12]
Related Work - SEFE and Beyond Planarity

- quasiplanar GSE
  - a tree and a cycle
  - a tree and an outerpillar

[Didimo et al. ’12]
[Di Giacomo et al. ’15]
Related Work - SEFE and Beyond Planarity

- quasiplanar GSE
  - a tree and a cycle [Didimo et al. ’12]
  - a tree and an outerpillar [Di Giacomo et al. ’15]
  - not every two quasiplanar graphs [Di Giacomo et al. ’15]
Related Work - SEFE and Beyond Planarity

- quasiplanar GSE
  - a tree and a cycle [Didimo et al. '12]
  - a tree and an outerpillar [Di Giacomo et al. '15]
  - not every two quasiplanar graphs [Di Giacomo et al. '15]
- simultaneous RAC drawings
  - [Argyriou et al. '13, Bekos et al. '16, Evans et al. '16, Grilli '18]
Our Results

- always positive instances for QuaSEFE
Our Results

- always positive instances for QuaSEFE
  - two planar graphs and a tree
Our Results

- always positive instances for QuaSEFE
  - two planar graphs and a tree
  - a 1-planar graph and a planar graph
Our Results

- always positive instances for QuaSEFE
  - two planar graphs and a tree
  - a 1-planar graph and a planar graph
  - planar graphs with restrictions on their intersection graphs
Our Results

- always positive instances for QuaSEFE
  - two planar graphs and a tree
  - a 1-planar graph and a planar graph
  - planar graphs with restrictions on their intersection graphs
- counterexamples for QuaSEFE in two special settings
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
Two Planar Graphs and a Tree ✓

- 1. Draw $G_1$ planar
- 2. Draw $T_2$ planar
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
Two Planar Graphs and a Tree ✓

- 1. Draw \(G_1\) planar
- 2. Draw \(T_2\) planar
  - some edges fixed by \(G_1\)
  - choose planar rotation system from \(G_3\) for edges in \(G_3 \setminus G_1\)
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
3. Draw $G_3$ quasiplanar
Two Planar Graphs and a Tree

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
3. Draw $G_3$ quasiplanar
   - $G_3 \setminus G_1$ planar
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
3. Draw $G_3$ quasiplanar
   - $G_3 \setminus G_1$ planar
Two Planar Graphs and a Tree

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
3. Draw $G_3$ quasiplanar
   - $G_3 \setminus G_1$ planar
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
3. Draw $G_3$ quasiplanar
   - $G_3 \setminus G_1$ planar
   - thickness 2 $\Rightarrow$ quasiplanar
Two Planar Graphs and a Tree ✓

1. Draw $G_1$ planar
2. Draw $T_2$ planar
   - some edges fixed by $G_1$
   - choose planar rotation system from $G_3$ for edges in $G_3 \setminus G_1$
   - remaining edges embedded planar
3. Draw $G_3$ quasiplanar
   - $G_3 \setminus G_1$ planar
   - thickness 2 $\Rightarrow$ quasiplanar
A 1-Planar Graph and a Planar Graph ✓

1. Decompose the 1-planar graph into a planar graph $G_1$ and a forest $T_2$ [Ackerman '14]
A 1-Planar Graph and a Planar Graph ✓

1. Decompose the 1-planar graph into a planar graph $G_1$ and a forest $T_2$ [Ackerman '14]

2. Apply the previous result ($G_1$ and $T_2$ are planar)
Triples of Planar Graphs ✓

- Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$
Triples of Planar Graphs ✓

Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$
Triples of Planar Graphs ✓

- Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$

- **Theorem**: If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.
Triples of Planar Graphs

Let $G_1$, $G_2$, and $G_3$ be planar graphs on $V$.

**Theorem:** If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.
Triples of Planar Graphs ✓

Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$

Theorem: If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.

$H_1, H_2, H_3, H_{1,2}, H_{1,3}, H_{2,3}$
Triples of Planar Graphs ✓

- Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$

- **Theorem**: If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.
Triples of Planar Graphs ✓

Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$

Theorem: If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.
Triples of Planar Graphs ✓

- Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$

- **Theorem:** If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.

- **Corollary:** $H_1 = \emptyset \Rightarrow$ QuaSEFE
Triples of Planar Graphs ✓

Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$.

Theorem: If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.

Corollary: $H_1 = \emptyset$ $\Rightarrow$ QuaSEFE
Triples of Planar Graphs

Let $G_1$, $G_2$ and $G_3$ planar graphs on $V$

**Theorem:** If $\langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle$ admits a SEFE, $\langle G_1, G_2, G_3 \rangle$ admits a QuaSEFE.

**Corollary:** $H_1 = \emptyset \Rightarrow$ QuaSEFE

**Corollary:** $H_{1,2}$ is forest of paths $\Rightarrow$ QuaSEFE
Triples of Planar Graphs

Let \( G_1, G_2 \) and \( G_3 \) planar graphs on \( V \).

**Theorem:** If \( \langle G_1 \setminus G_3, G_2 \setminus G_3 \rangle \) admits a SEFE, \( \langle G_1, G_2, G_3 \rangle \) admits a QuaSEFE.

**Corollary:** \( H_1 = \emptyset \) \( \Rightarrow \) QuaSEFE

**Corollary:** \( H_{1,2} \) is forest of paths \( \Rightarrow \) QuaSEFE

**Theorem:** If \( H \) is a forest of paths, \( \langle G_1, G_2, G_3 \rangle \) admits a QuaSEFE.
Sunflower Instances ✓

- **Sunflower Instance**: Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
Sunflower Instances ✓

- **Sunflower Instance**: Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
  - Deciding if SEFE exists is NP-hard [Angelini et al. '15]
  - for $k \geq 3$ [Schaefer '13]
Sunflower Instances ✓

- **Sunflower Instance**: Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
  - Deciding if SEFE exists is NP-hard [Angelini et al. ’15] for $k \geq 3$
  - **Corollary**: A sunflower instance with $k = 3$ planar graphs admits a QuaSEFE.
Sunflower Instances ✓

- **Sunflower Instance**: Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
  - Deciding if SEFE exists is NP-hard [Angelini et al. ’15] [Schaefer ’13] for $k \geq 3$
  - **Corollary**: A sunflower instance with $k = 3$ planar graphs admits a QuaSEFE.
- **Theorem**: For any $k$, a sunflower instance with $k$ planar graphs admits a QuaSEFE.
Sunflower Instances ✓

- **Sunflower Instance:** Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
  - Deciding if SEFE exists is NP-hard \cite{Angelini et al. '15} for $k \geq 3$
  - **Corollary:** A sunflower instance with $k = 3$ planar graphs admits a QuaSEFE.

- **Theorem:** For any $k$, a sunflower instance with $k$ planar graphs admits a QuaSEFE.
  - 1. Draw $H$ planar
**Sunflower Instances ✓**

- *Sunflower Instance*: Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
  - Deciding if SEFE exists is NP-hard \cite{Angelini15} for $k \geq 3$
  - **Corollary**: A sunflower instance with $k = 3$ planar graphs admits a QuaSEFE.

- **Theorem**: For any $k$, a sunflower instance with $k$ planar graphs admits a QuaSEFE.
  - 1. Draw $H$ planar
  - 2. Draw each $G_i \setminus H$ planar
Sunflower Instances ✓

- **Sunflower Instance**: Planar Graphs $G_1, \ldots, G_k$ s.t. each edge is either in exactly one $G_i$ or in all $G_i$ (i.e. in $H := \bigcap G_i$)
  - Deciding if SEFE exists is NP-hard for $k \geq 3$ [Angelini et al. '15]
  [Schaefer '13]
- **Corollary**: A sunflower instance with $k = 3$ planar graphs admits a QuaSEFE.
- **Theorem**: For any $k$, a sunflower instance with $k$ planar graphs admits a QuaSEFE.
  1. Draw $H$ planar
  2. Draw each $G_i \setminus H$ planar
  - each $G_i$ is drawn with thickness 2
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$. 
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$. 

![Diagram of a graph with labeled vertices](image-url)
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$. 

$M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\}$
**Theorem:** There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

$M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\}$
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

\[ M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\} \]
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

$M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\}$
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

$$M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\}$$
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

$M_2 = M_1 \setminus \{ (v_{17}, v_{18}), (v_{19}, v_{20}) \} \cup \{ (v_{18}, v_{20}) \}$
Theorem: There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

\[ M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\} \]
**Theorem:** There are two matchings $M_1$ and $M_2$ that do not admit a QuaSEFE for a fixed drawing of $M_1$.

$$M_2 = M_1 \setminus \{(v_{17}, v_{18}), (v_{19}, v_{20})\} \cup \{(v_{18}, v_{20})\}$$
Open Problems

- Do the following always admit a QuaSEFE?
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
  - three outerplanar graphs
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
  - three outerplanar graphs
  - four paths
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
  - three outerplanar graphs
  - four paths

- What is the computational complexity of QuaSEFE?
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
  - three outerplanar graphs
  - four paths
- What is the computational complexity of QuaSEFE?
- Extend to other beyond planar graph classes such as $k$-planar graphs.
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
  - three outerplanar graphs
  - four paths
- What is the computational complexity of QuaSEFE?
- Extend to other beyond planar graph classes such as $k$-planar graphs.
  - Main difficulty: find a similarly catchy name for the problem
Open Problems

- Do the following always admit a QuaSEFE?
  - two 1-planar graphs
  - a quasiplanar graph and a matching
  - three outerplanar graphs
  - four paths
- What is the computational complexity of QuaSEFE?
- Extend to other beyond planar graph classes such as $k$-planar graphs.
  - Main difficulty: find a similarly catchy name for the problem

Thank you for your attention!