COLORING HASSE DIAGRAMS AND DISJOINTNESS GRAPHS OF CURVES

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Chromatic Number vs. Clique Number

$\chi(G)$  chromatic number - minimum number of colors needed to color $V(G)$ so that no edge is monochromatic

$\omega(G)$  clique number - maximum size of a complete subgraph of $G$

Theorem (Erdős 1959)
For every $k$ and $\ell$, there exists a graph $G = G(k, \ell)$ with $\chi(G) = k$ and with no cycle of length $\leq \ell$. 
**Theorem (Gallai, Hajós)**
For the intersection graph of any system of intervals along a line, we have $\chi(G) = \omega(G)$.

\[ \begin{align*}
\omega &= 2 \\
\chi &= 3
\end{align*} \]
**X-bounded Families of Graphs**

A family of graphs $G$ is $X$-bounded if there exists a function $f$ such that $X(G) \leq f(\omega(G))$ for all $G \in G$ (Gyárfás - Lehel 1985)

**Theorem (Asplund - Grünbaum 1960)**

The family of intersection graphs of axis-parallel rectangles in the plane is $X$-bounded.
**X-bounded Families of Graphs**

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**Theorem (Asplund-Grünbaum 1960)**
The family of intersection graphs of axis-parallel rectangles in the plane is \( X \)-bounded.

The disjointness graph of a family of objects is the complement of its intersection graph.

**Theorem**
The family of disjointness graphs of axis-parallel rectangles in the plane is \( X \)-bounded.

\[ X(G) \leq c \cdot \omega(G) \]
Not \( X \)-bounded Families of Graphs

Theorem (Pawlik-Kozik-Krawczyk-Lason-Micek-Trotter-Walczak 2014)

There exist triangle-free \((\omega = 2)\) intersection graphs of curves in the plane with arbitrarily large chromatic numbers \(X\). \[\Omega(\log \log n)\]
Not $\chi$-bounded Families of Graphs

Theorem (Pawlik-Kozik-Krawczyk-Lasoń-Micek-Trotter-Walczak 2014)
There exist triangle-free ($\omega=2$) intersection graphs of curves in the plane with arbitrarily large chromatic numbers $\chi$. $[\Omega(\log \log n)]$

Theorem (P.-Tardos-Tóth 2017)
There exist triangle-free ($\omega=2$) disjointness graphs of curves in the plane with arbitrarily large chromatic numbers $\chi$. $[[\log_2 n]]$
Grounded Curves and Cover Graphs

Grounded curves on the y-axis

→

Disjointness graph

1 2 3 4
**Grounded Curves and Cover Graphs**

- Grounded curves on the y-axis
- Disjointness graph
- $b$ covers $a$ if $b > a$ and there is no $c \in P$ with $b > c > a$
- $ba$ is an edge of the cover graph (= undirected Hasse diagram)
Theorem (Sinden 1966 ↔, Middendorf-Pfeiffer 1993 →)

G is a cover graph ⇔

- G is triangle-free and
- G is the disjointness graph of a family of grounded curves.

\[ (P, \prec) \] partial order
Theorem (Sinden 1966 ↔, Middendorf-Pfeiffer 1993 →)

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Theorem (P.-Tomon 2019)
For every r and n, there exists a partially ordered set of n elements whose cover graph has girth \geq r and chromatic number \geq \Omega \left( \frac{1}{r} \log n \right).
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Theorem (P.-Tomon 2019)
For every r and n, there exists a partially ordered set of n elements whose cover graph has girth \( \geq r \) and chromatic number \( \geq \Omega \left( \frac{1}{r} \log n \right) \).

Corollary
For every r and n, there exists a family of n curves whose disjointness graph has girth \( \geq r \) and chromatic number \( \geq \Omega \left( \frac{1}{r} \log n \right) \). \[ \rightarrow r=4 \text{ case] } \]
**Uniquely Generated Posets**

For every \( x < y \), there is a unique path \( x = v_1 \uparrow v_2 \uparrow \ldots \uparrow v_k \), where \( v_{i+1} \) covers \( v_i \).
**Uniquely Generated Posets**

For every $x < y$, there is a unique path $x = v_1 \prec v_2 \prec \ldots \prec v_k$, where $v_{i+1}$ covers $v_i$.

**Theorem (P.-Tomon 2019)**

(i) If $P$ is a uniquely generated poset with $n$ vertices, then for its cover graph $G$, we have $\chi(G) \leq \lceil \log_2 n \rceil + 1$.

(ii) For every $r > 3$ and $n > n_0(r)$, there exists a uniquely generated poset with $n$ vertices whose cover graph $G$ has girth $\geq r$ and $\chi(G) \geq \Omega \left( \frac{1}{r} \log_2 n \right)$. 
Theorem (P.-Tomon 2019)

(i) If \( P \) is a uniquely generated poset with \( n \) vertices, then for its cover graph \( G \), we have \( \chi(G) \leq \lceil \log_2 n \rceil + 1 \).

Proof. Use greedy coloring with 1, 2, 3, ...
Let \( T(v) = \{ u \in P : u < v \} \) (tree)

\[
|T(v)| \geq 1 + \sum_{i=1}^{k-1} |T(u_i)|
\]

\[
\geq 1 + \sum_{i=1}^{k-1} 2^{i-1}
\]

\[
= 2^{k-1}
\]
Theorem.

(ii) For every $r > 3$ and $n > n_0(r)$, there exists a uniquely generated poset with $n$ vertices whose cover graph $G$ has girth $\geq r$ and $\chi(G) = \Omega(\frac{1}{r} \log_2 n)$.

Proof. Let $k = \frac{\log n}{10r}$, $m = \frac{n}{k} = 10r \frac{n}{\log n}$

$$|A_1| = \ldots = |A_k| = m$$

$$P_{ij} = \frac{2^{j-i}}{m} < \frac{1}{10r} \frac{\log n}{n^{1-1/10r}}$$
Proof. Let \( k = \frac{\log n}{10r} \), \( m = \frac{n}{k} = 10r \frac{n}{\log n} \)

\[ |A_i| = \ldots = |A_k| = m \]

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Proof. Let $k = \frac{\log n}{10r}$, $m = n/k = 10r\frac{n}{\log n}$

$G \cdots \cdots A_i \cdots \cdots A_j \cdots \cdots A_k \cdots \cdots$

$|A_i| = \ldots = |A_k| = m$

$P_{ij} = \frac{2^{j-i}}{m} < \frac{1}{10r} \frac{\log n}{n^{1-1/10r}}$

Claim. With positive probability $G$ satisfies

1. there is no independent set of size $m$,
2. there are $\leq \frac{n}{3}$ cycles of length $< r$,
3. there are $\leq \frac{n}{3}$ pairs $u, v \in V(G)$ with 2 edge-disjoint monotone paths connecting them.
Proof. Let \( k = \frac{\log n}{10r} \), \( m = \frac{n}{k} = 10r \frac{n}{\log n} \)

\[
G \quad A_1 \ldots A_i \ldots A_j \ldots A_k \quad |A_i| = \ldots = |A_k| = m
\]

\[
P_{ij} = \frac{2^{j-i}}{m} < \frac{1}{10r} \frac{\log n}{n^{1-1/10r}}
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Claim. With positive probability \( G \) satisfies
(1) there is no independent set of size \( m \),
(2) there are \( \leq \frac{n}{3} \) cycles of length \( < r \),
(3) there are \( \leq \frac{n}{3} \) pairs \( u, v \in V(G) \) with 2 edge-disjoint monotone paths connecting them.

Then
- delete a vertex from each "bad" cycle + pair,
- define \( u < v \) if there is a monotone increasing path from \( u \) to \( v \).
Problem. Determine or estimate $k(n)$, the maximum chromatic number of the cover graph of an $n$-element partially ordered set.

\[ k(n) = O(\sqrt{n}) \]
\[ k(n) = O(\sqrt{n/\log n}) \quad \text{Ajtai-Komlós-Szemeredi 1980} \]
\[ k(n) = \Omega(\log n / \log \log \log n) \quad \text{Brightwell-Nešetřil 1991} \]
\[ k(n) = \Omega(\log n) \quad \text{P.-Tamon 2019} \]