Graphs with large total angular resolution

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Definition

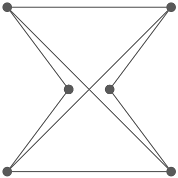
Definition (Total angular resolution)

The total angular resolution of a straight-line drawing is the minimum angle between two intersecting edges of the drawing.

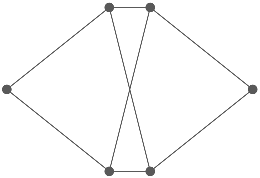
The total angular resolution of a graph G, or short $\mathrm{TAR}(G)$, is the maximum total angular resolution over all straight-line drawings of this graph.

Motivation

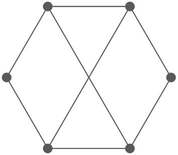
Crossing resolution



Angular resolution



Total angular resolution



Considered questions

- Can we find an upper bound for the number of edges of graphs G with $\text{TAR}(G) > 60^\circ$?
- What is the complexity of deciding whether $\text{TAR}(G) \geq 60^\circ$?

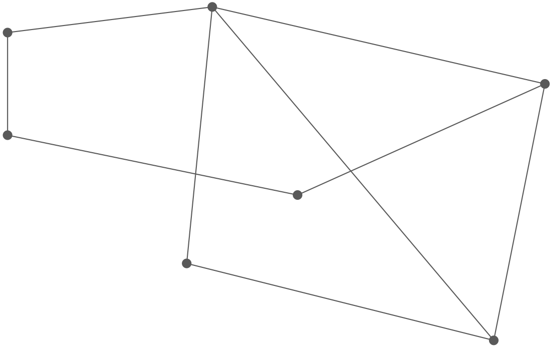
Upper bounds for the number of edges

Number of edges of drawings with:

- crossing resolution 90°: $\leq 4n - 10$
  [Didimo, Eades, Liotta, 2011]
- crossing resolution greater than 60°: $\leq 6.5n - 10$
  [Ackermann, Tardos, 2007]
- total angular resolution greater than 60°: $\leq 2n - 6$
  with some small exceptions
  [This work]

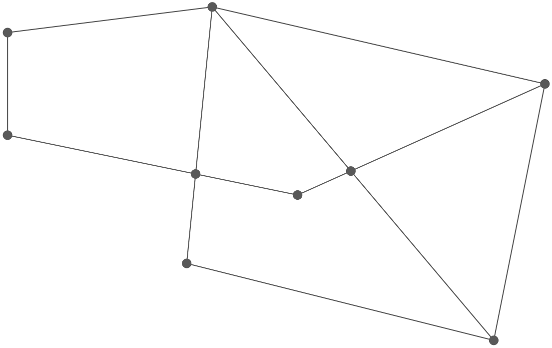
Planarized drawing

Planarized drawing: replace every crossing by a vertex.



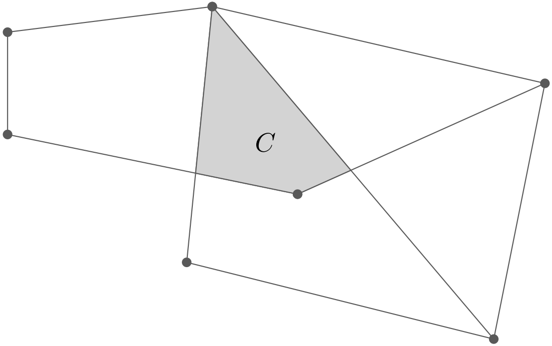
Planarized drawing

Planarized drawing: replace every crossing by a vertex.



Size of a cell

Size of a cell: number of sides in planarized drawing incident to this cell.



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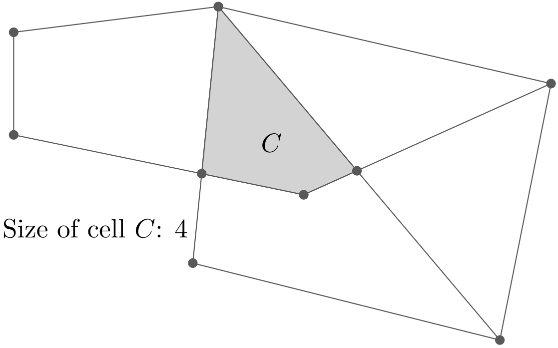

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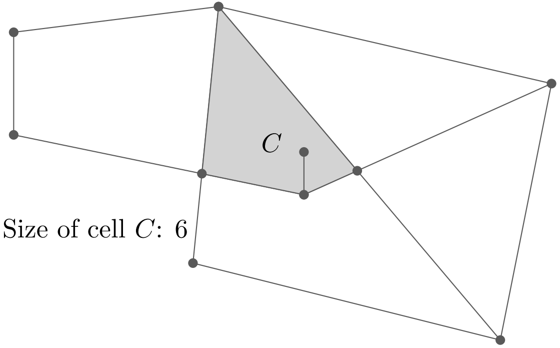

Basic idea

Let D be a drawing.
If $\text{TAR}(D) > 60^\circ$, then D does not contain a triangle and no three edges cross in one point.
So every cell has at least size 4.

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Lemma

Given a connected drawing D with $n \geq 1$ vertices and m edges. The unbounded cell of D has size k and $\mathrm{TAR}(D) > 60^\circ$. Then $m \leq 2n - 2 - \lceil k/2 \rceil$.

$m \leq 2n - 4$

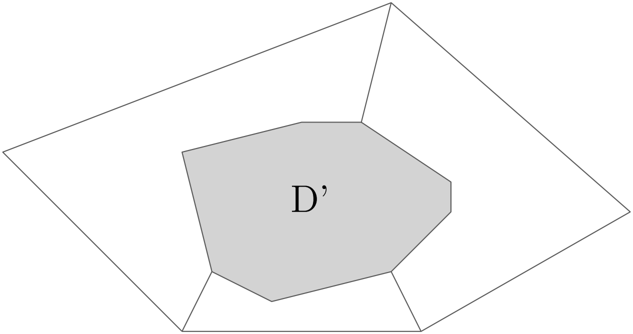
Lemma

Given a drawing D with $\mathrm{TAR}(D) > 60^\circ$. If the unbound cell has size at least 4, then $m \leq 2n - 4$.

The only possible triangle-free drawings with an unbound cell of size at most 2 are:

- the empty graph
- a single vertex
- two vertices joined by an edge.

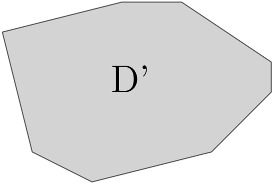
Idea to continue



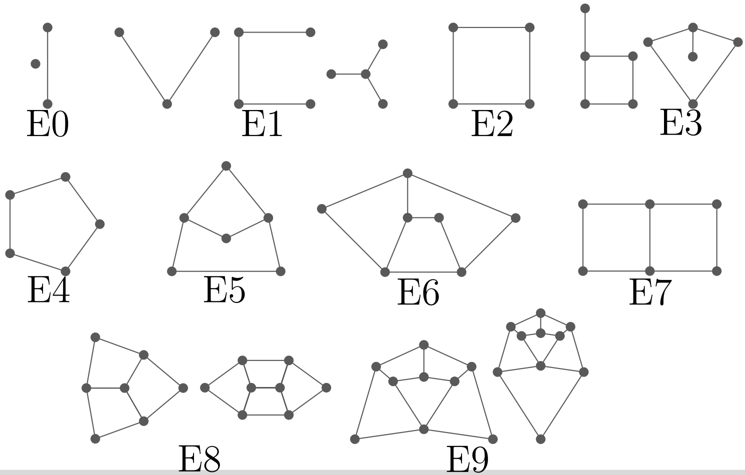
Idea to continue

$$m' \leq 2n' - 4$$
$$m' \geq m - 8$$
$$n' = n - 5$$

$$m \leq 2n - 6$$



Exceptions



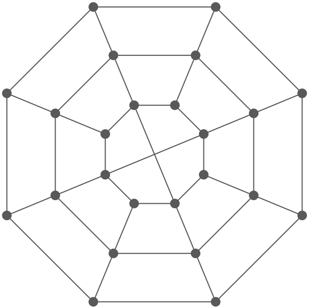
Result

Theorem

Given a graph G with $\mathrm{TAR}(G) > 60^\circ$.
Then $m \leq 2n - 6$ or G is in the exceptions.

Tightness

Drawing of a graph with $\text{TAR}(G) > 60^\circ$ and $2n - 6$ edges.



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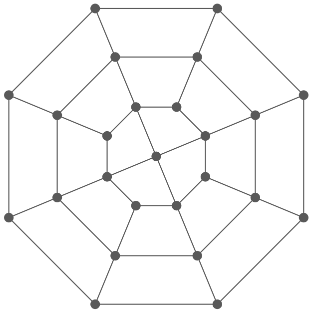

Hardness results

Before: It is NP-hard to decide whether a graph G has angular resolution $\geq 90^\circ$. [Forman et al. 1993]

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Theorem

It is NP*-hard to decide whether a graph G has* TAR(G) $\geq 60°$.

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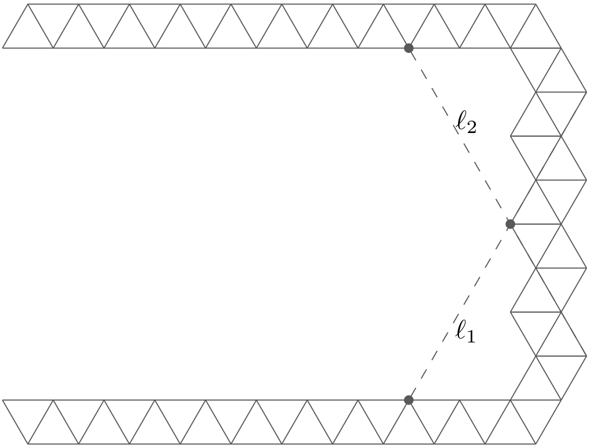
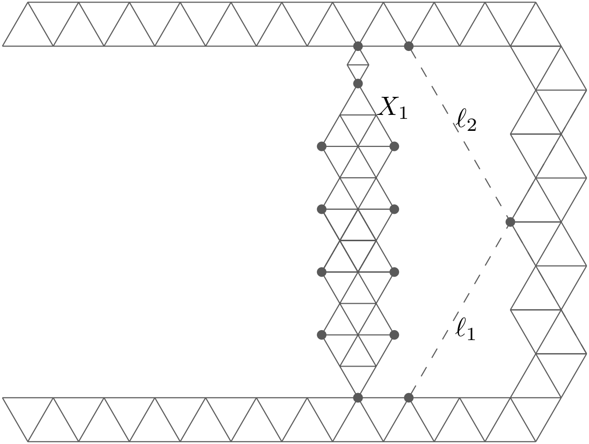
It is NP*-hard to decide whether a graph G has* TAR(G) $\geq 60^\circ$*.*

Proof by reduction from 3SAT.

Construction



Variable gadgets



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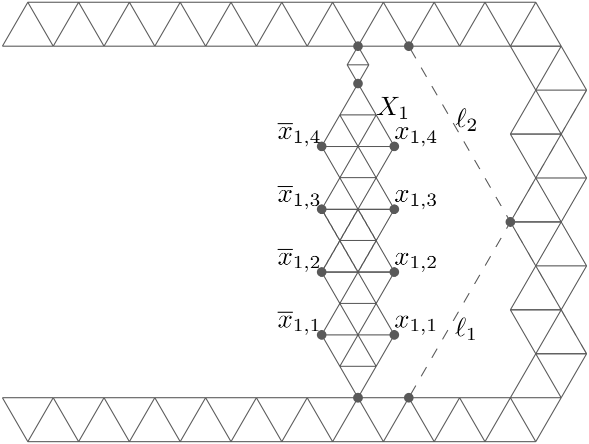

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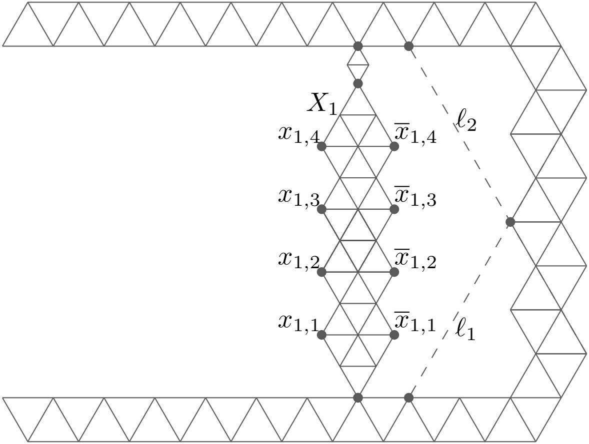
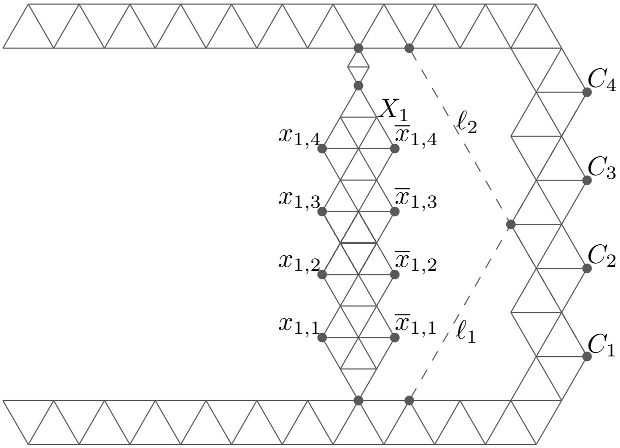


Clause gadget



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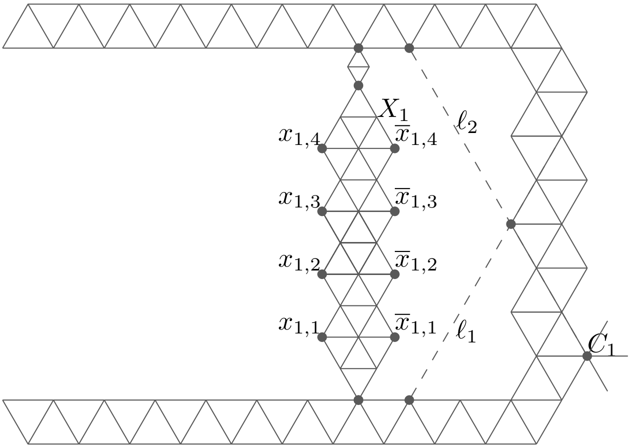
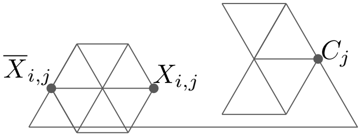


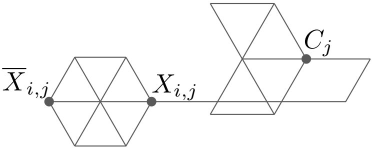
Connections

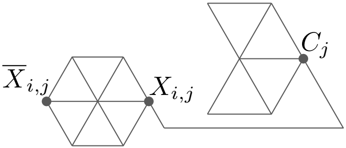
Connection to:

left side of variable gadget



right side of variable gadget





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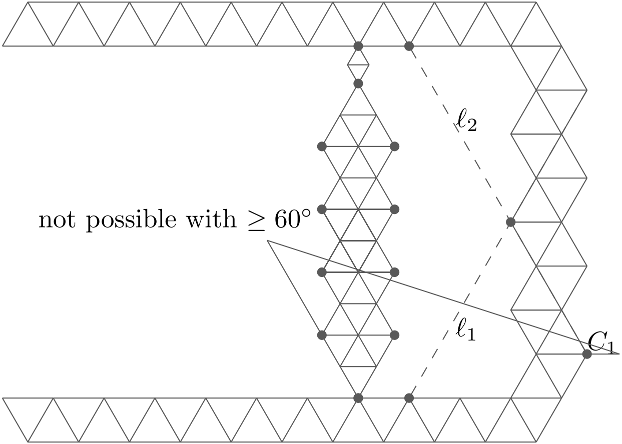

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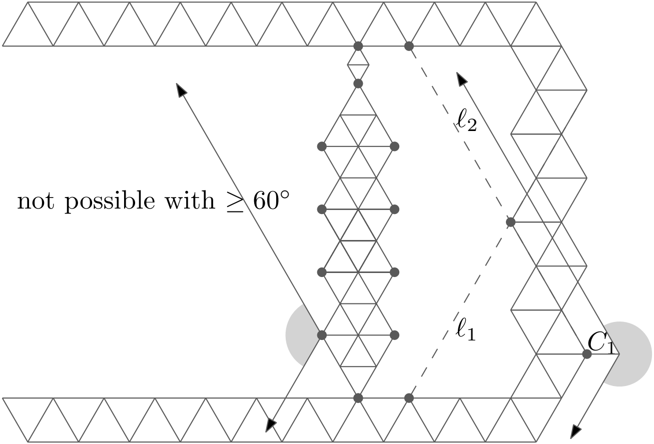
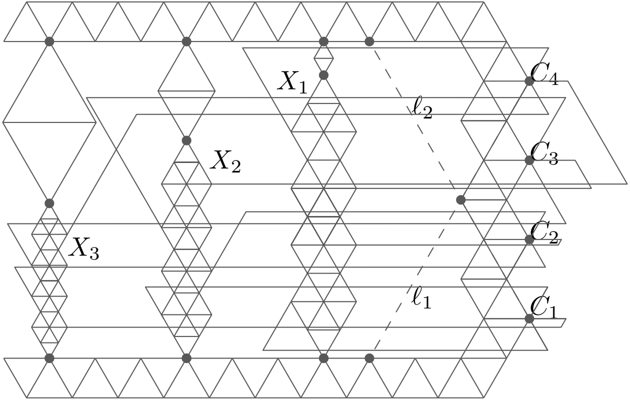


Example

$$(x_1 \vee \overline{x}_2 \vee x_3) \wedge (\overline{x}_1 \vee x_2 \vee \overline{x}_3) \wedge (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3) \wedge (x_1 \vee x_2 \vee x_3)$$



Open problems

Do almost all graphs with $\mathrm{TAR}(G) > \frac{k-2}{k}90^\circ$ have at most $2n-2-\lfloor \frac{k}{2} \rfloor$ edges?

At which angle(s) α does the decision problem, whether $\mathrm{TAR}(G) \geq \alpha$, change from NP-hard to polynomially solvable?