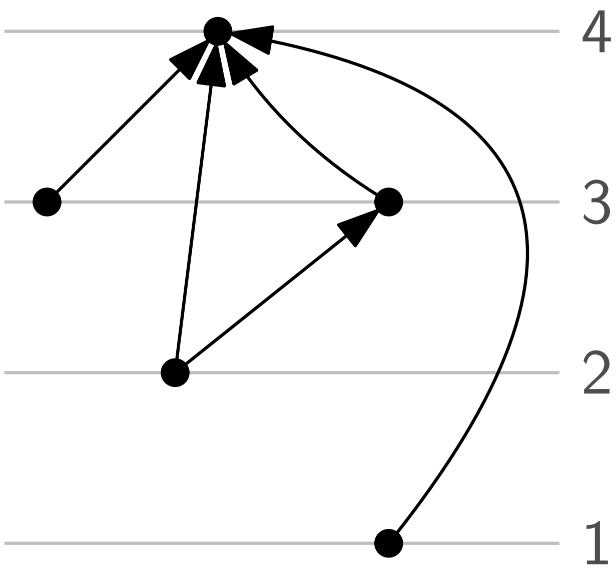
Level-Planar Drawings with Few Slopes

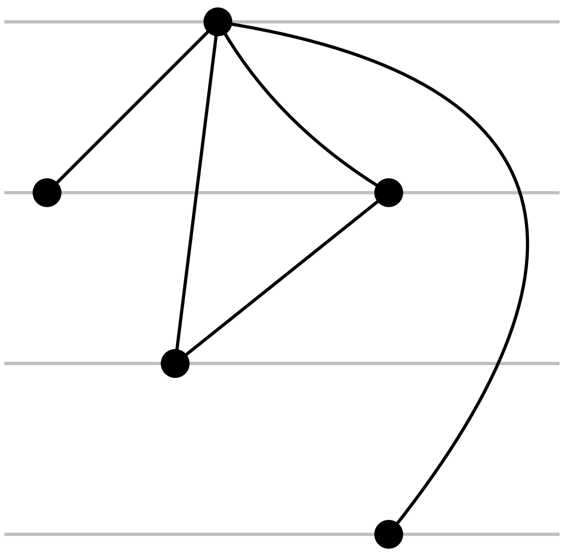
Guido Brückner Nadine Krisam Tamara Mchedlidze

Level Graphs



- directed graph $G = (V, E)$
- level assignment $\ell : V \to \mathbb{N}$ s.t. $\forall (u, v) \in E : \ell(u) < \ell(v)$

Level Graphs



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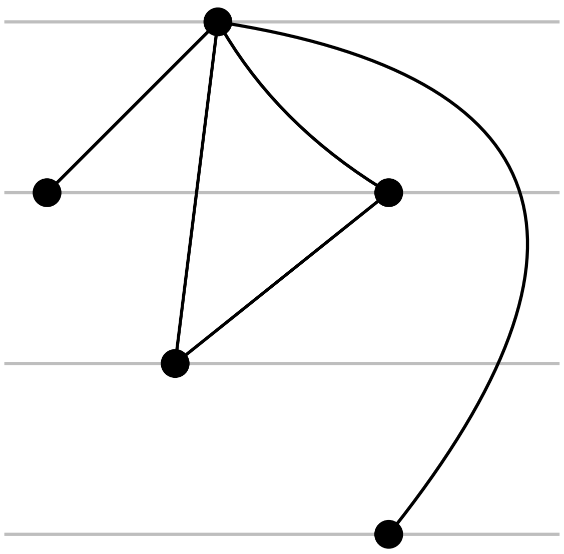

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  $\iff$ left-to-right order of the vertices on each level is fixed

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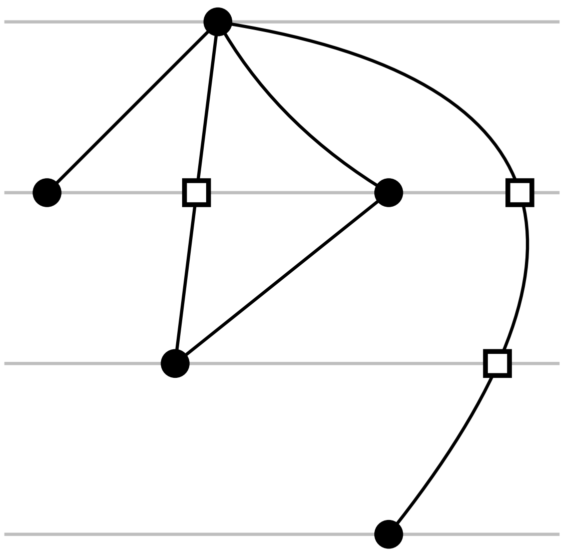

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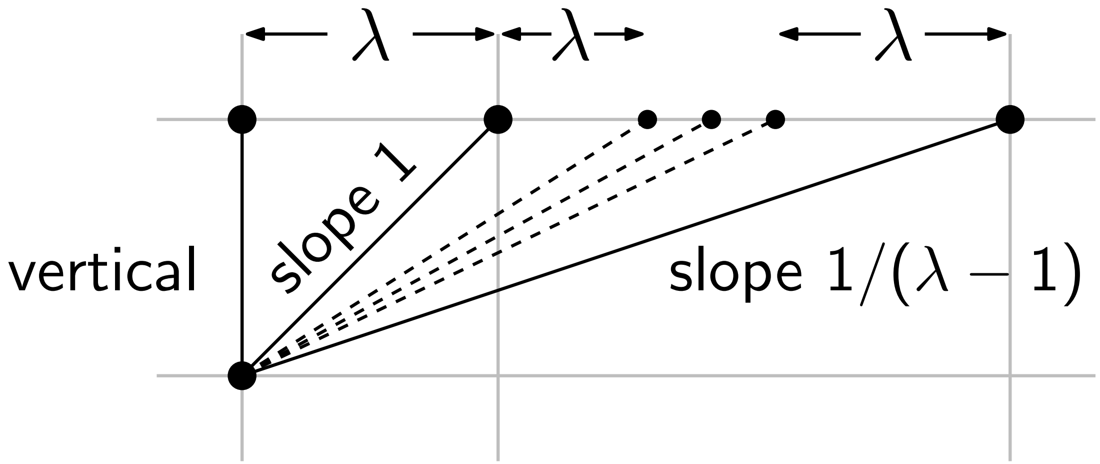
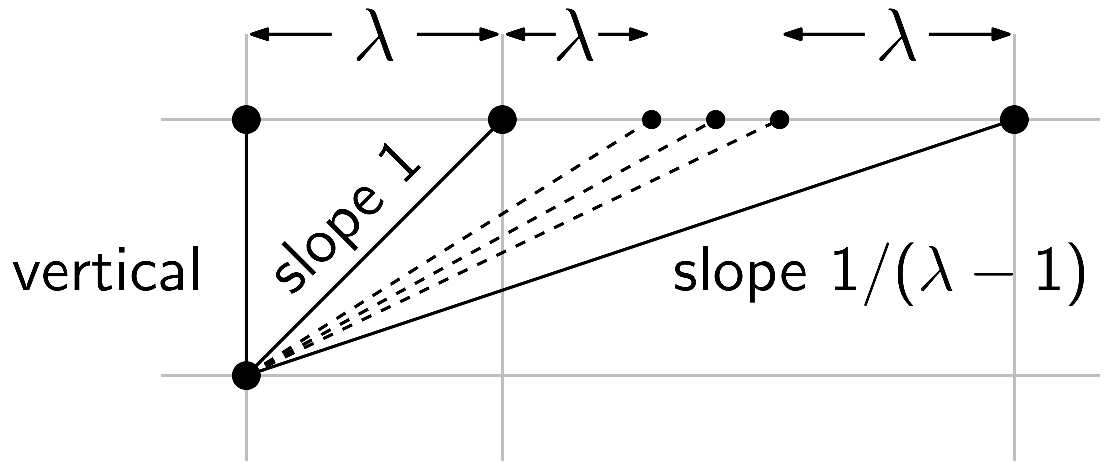
λ-Drawing Model

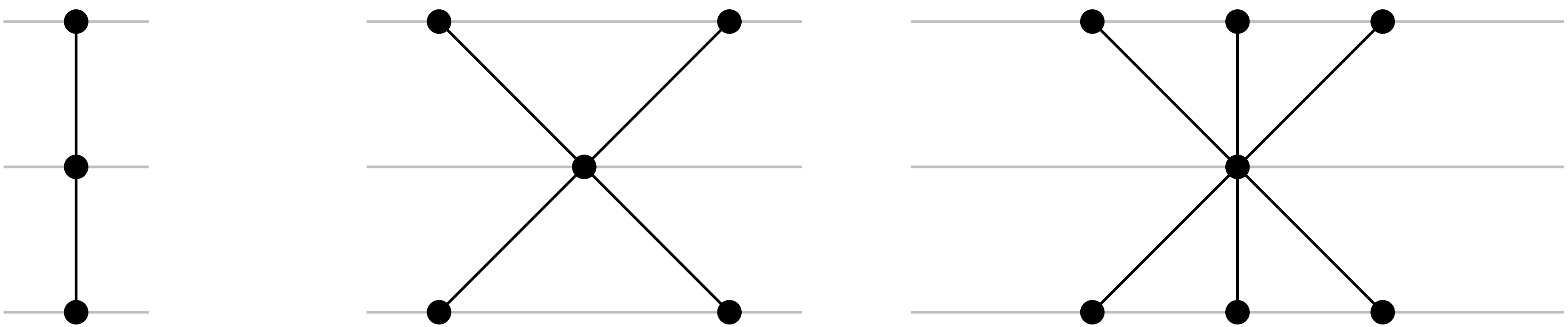
λ-drawing:



λ-Drawing Model

$\boldsymbol{\lambda}$-drawing:





1-drawing $\equiv$ 2-drawing $\equiv$ 3-drawing

λ-Drawing Model

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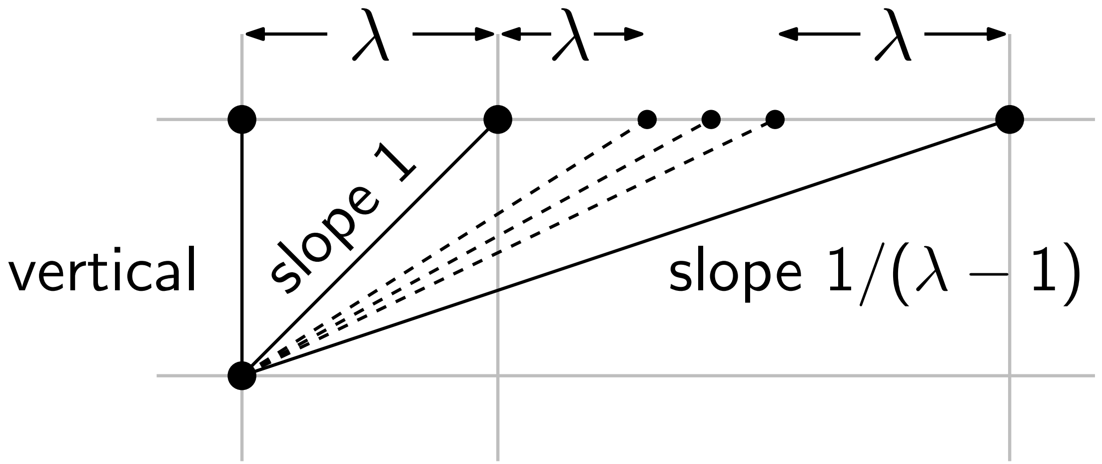

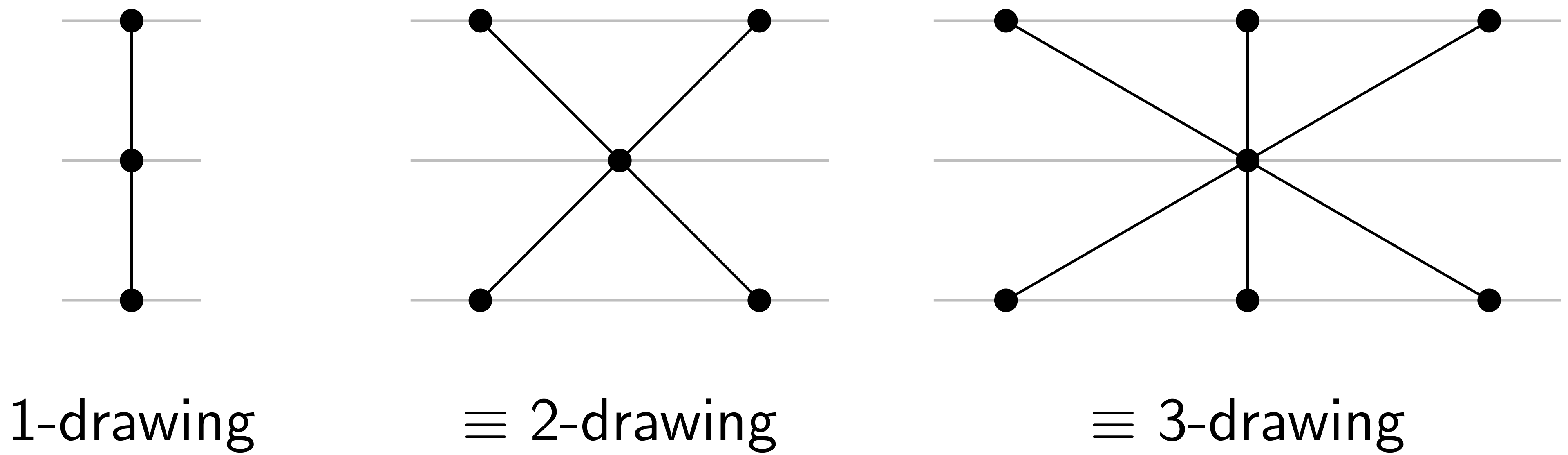

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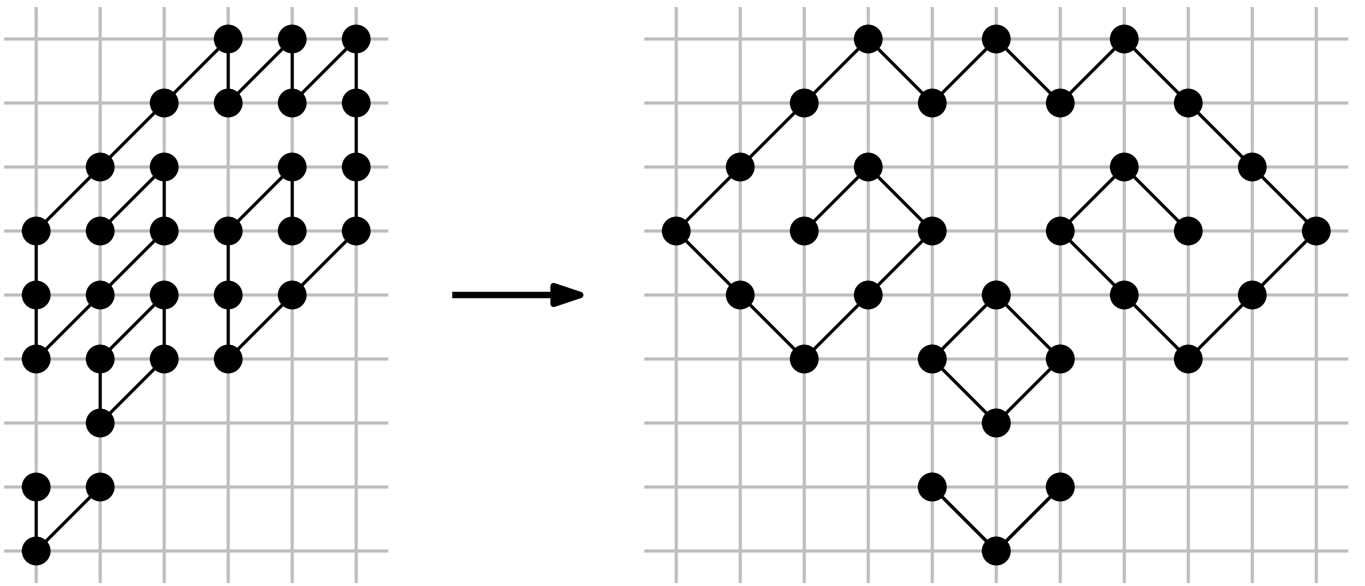
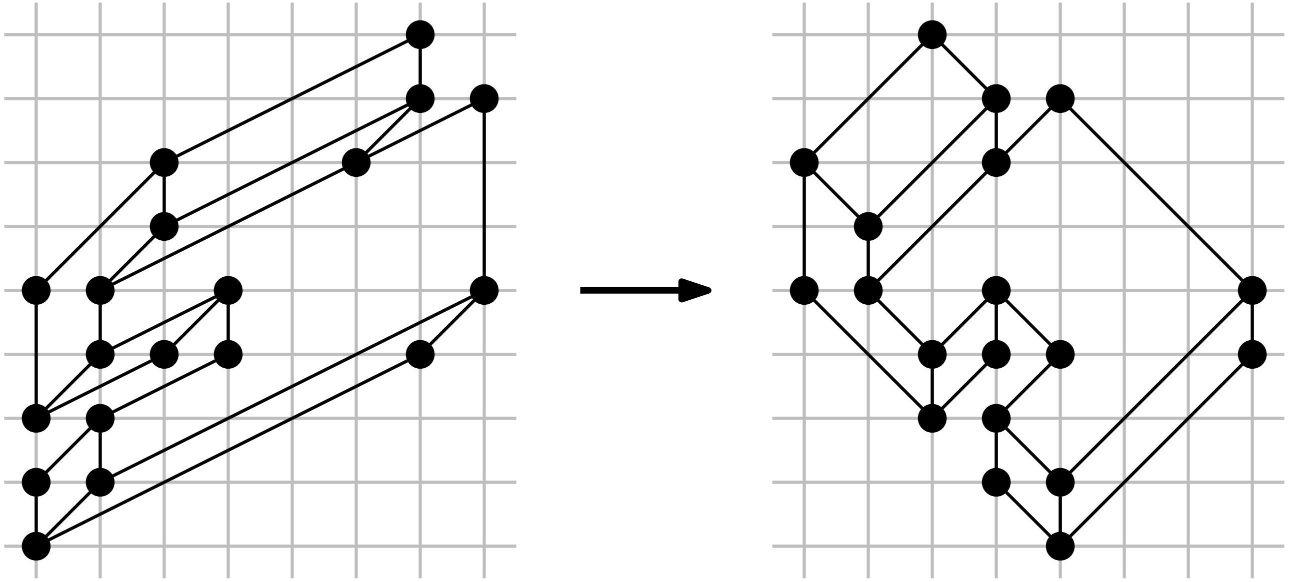
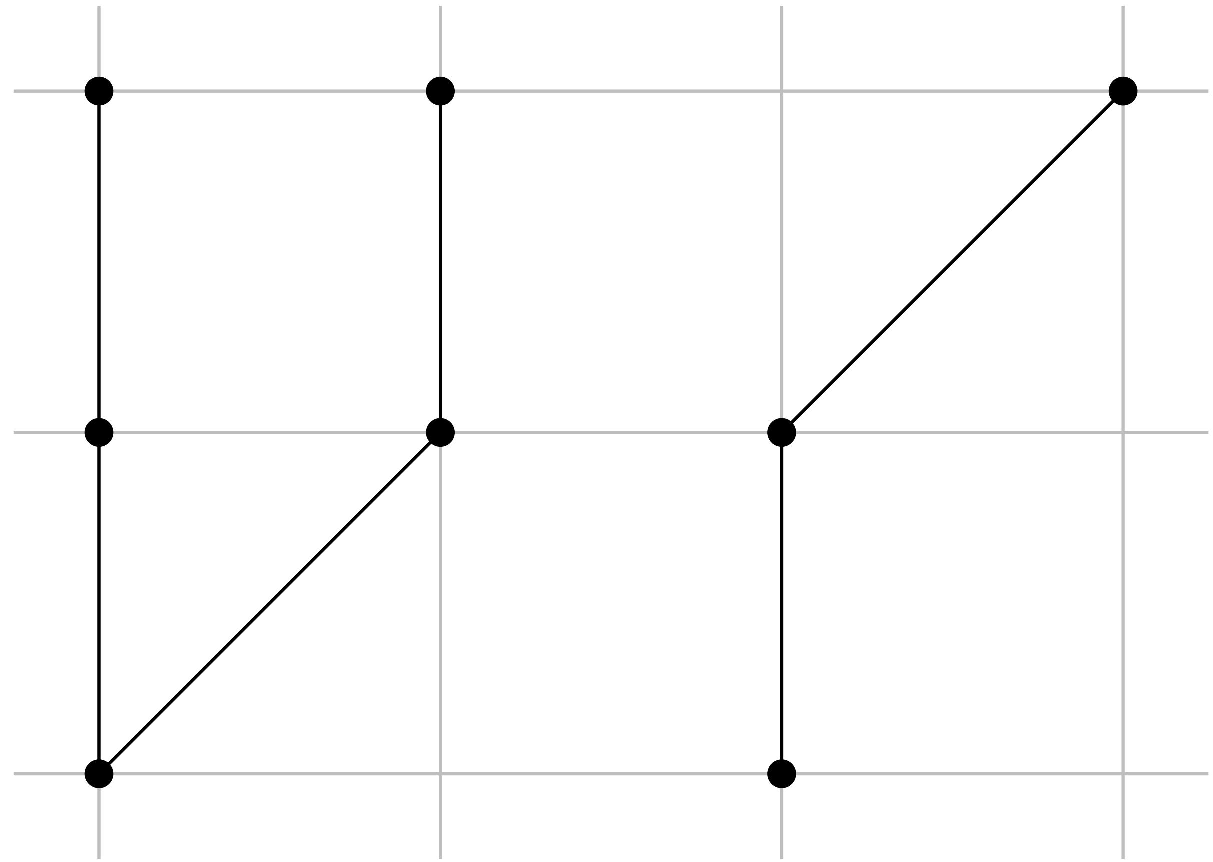




Flow Network



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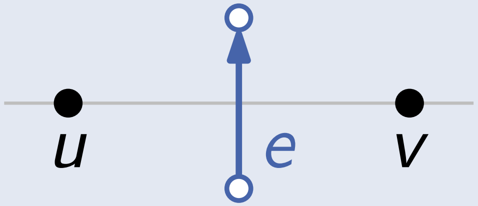
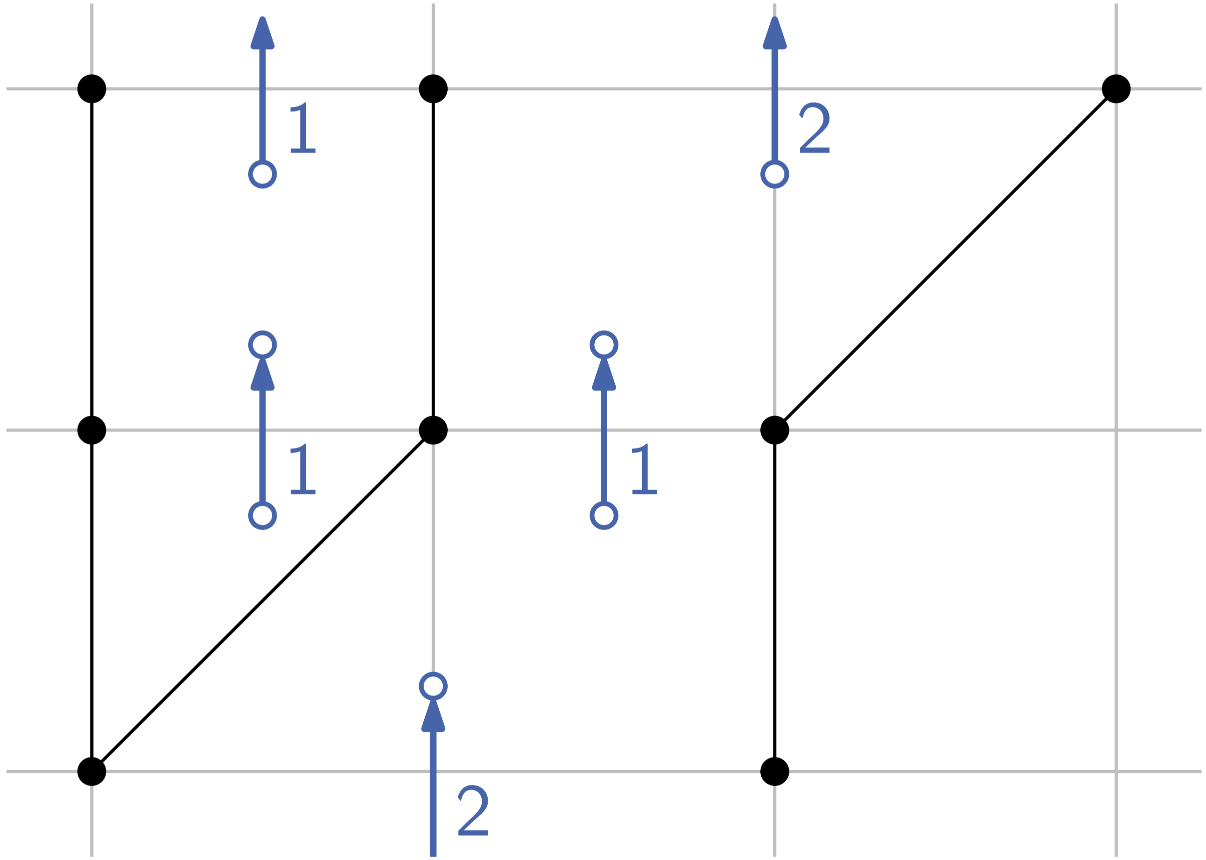


Flow on e = distance between u and v

Constraint $\varphi(e) \geq 1$



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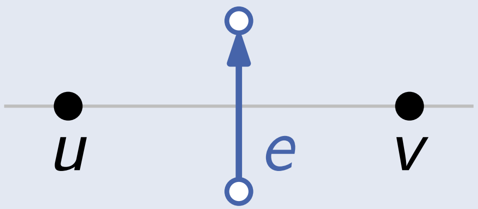

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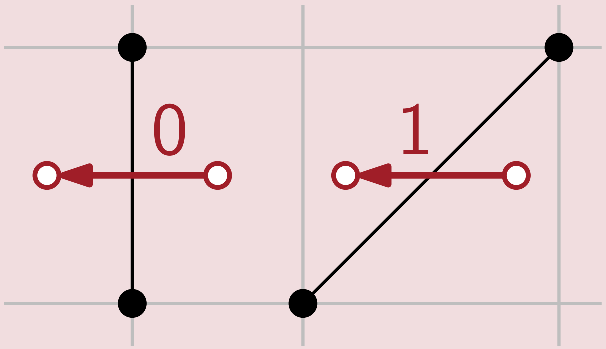
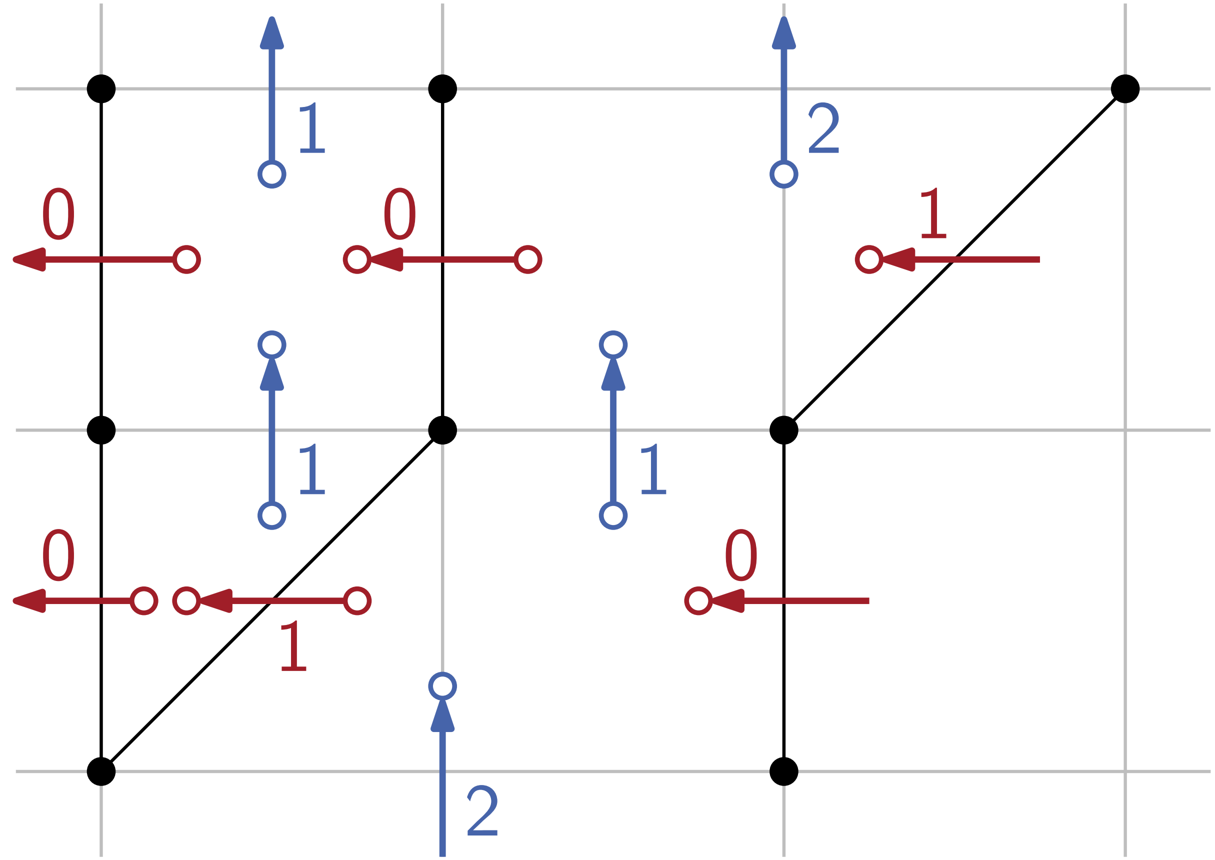


Flow = slope of dual edge

Constraint

$0 \leq \varphi(\cdot) \leq 1$



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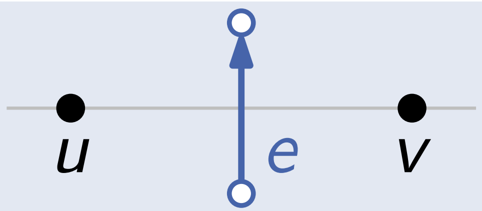

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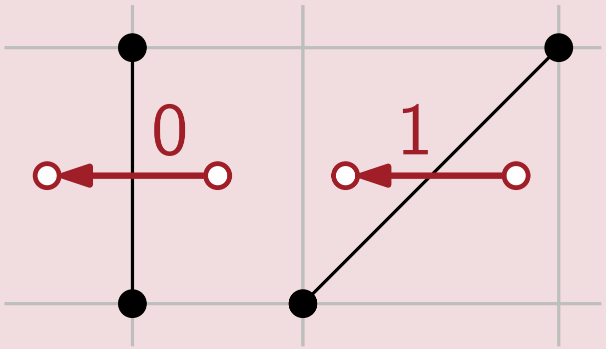

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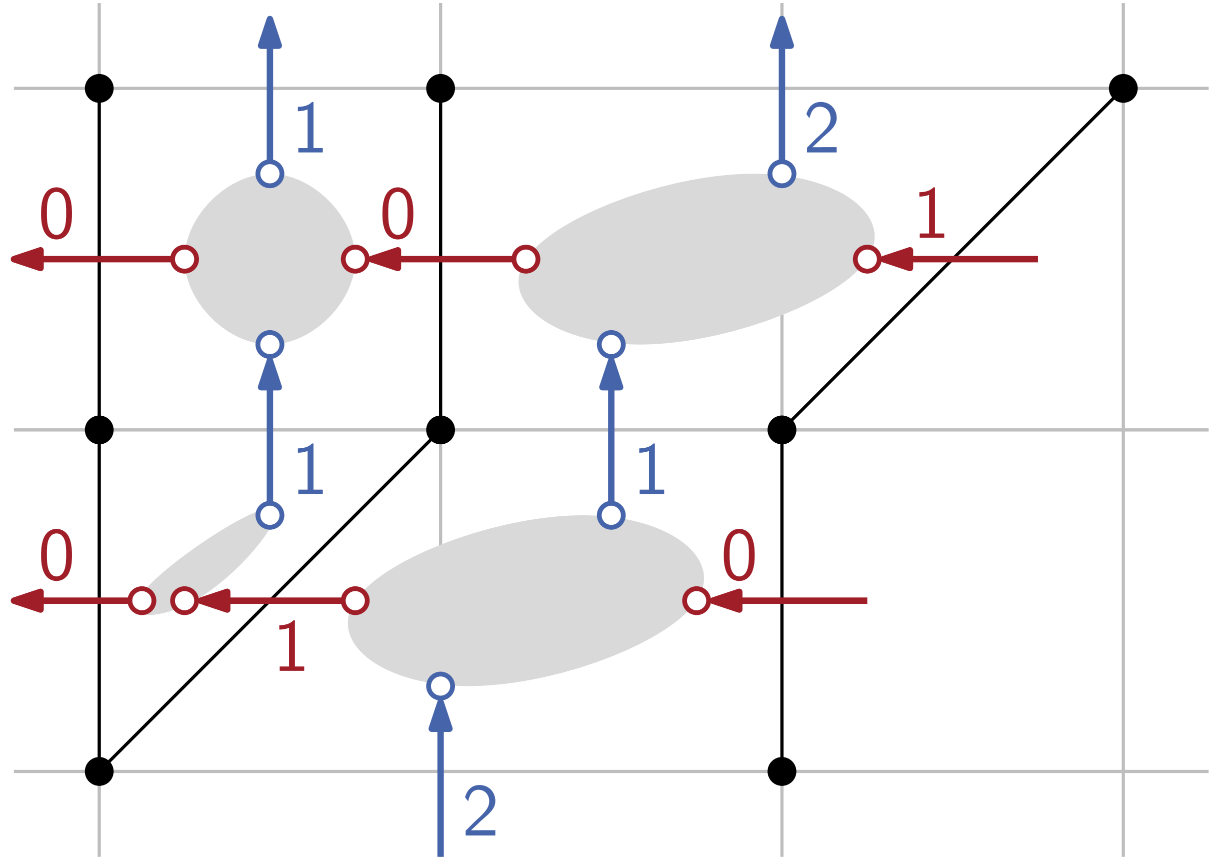
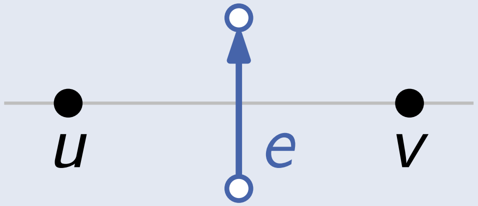


Lemma
Every admissible flow corresponds to a 2-slope drawing.

Flow Network



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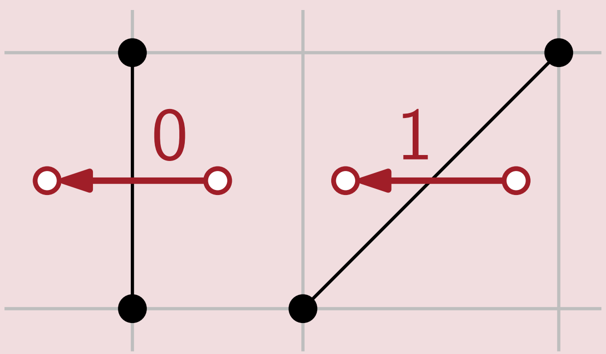

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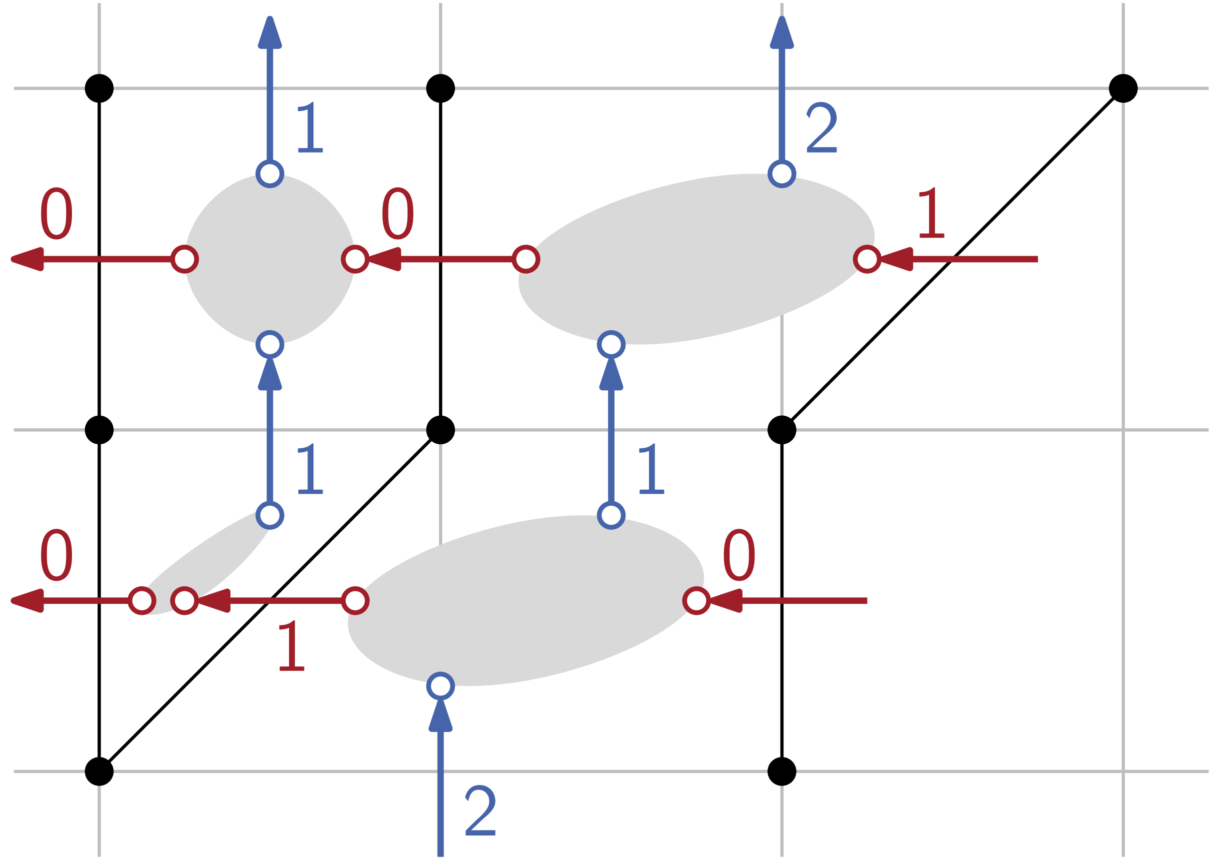

Lemma
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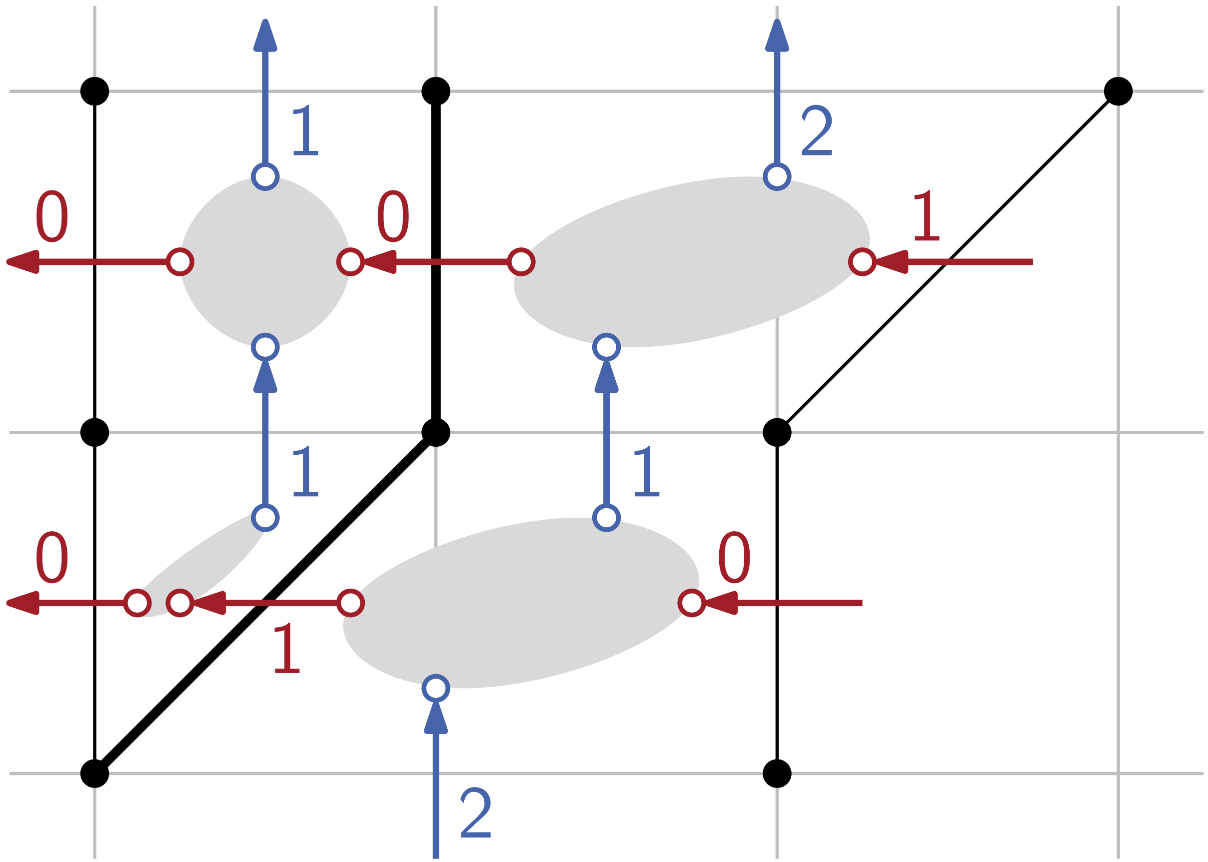
max-flow: $O(n \log^3 n)$

min-cost flow: $O(n^2 \log^2 n)$

Flow Network

Advanced Problems:

- partial drawing extension (simple in connected case)



Flow Network

Advanced Problems:

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  – real relaxation?

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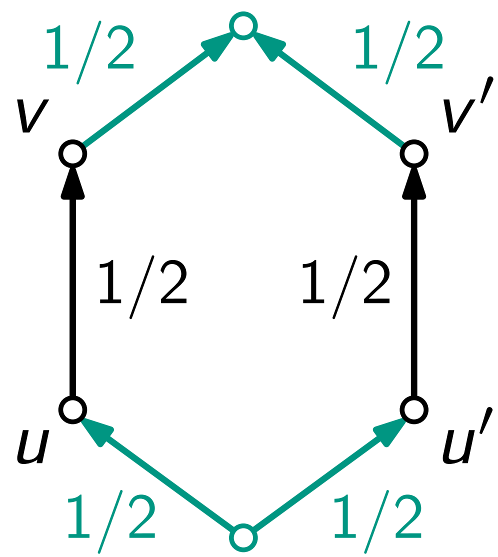

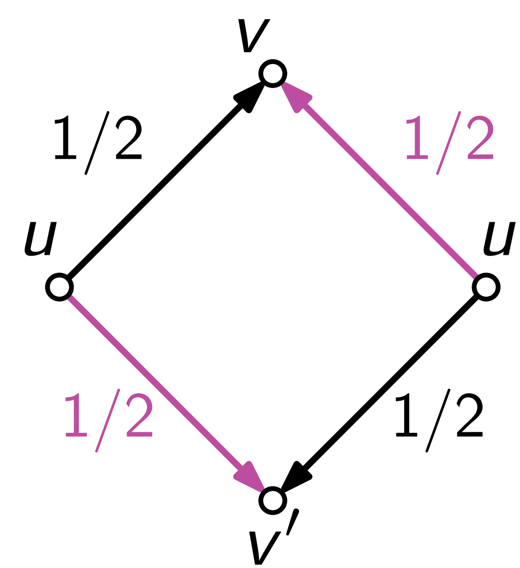

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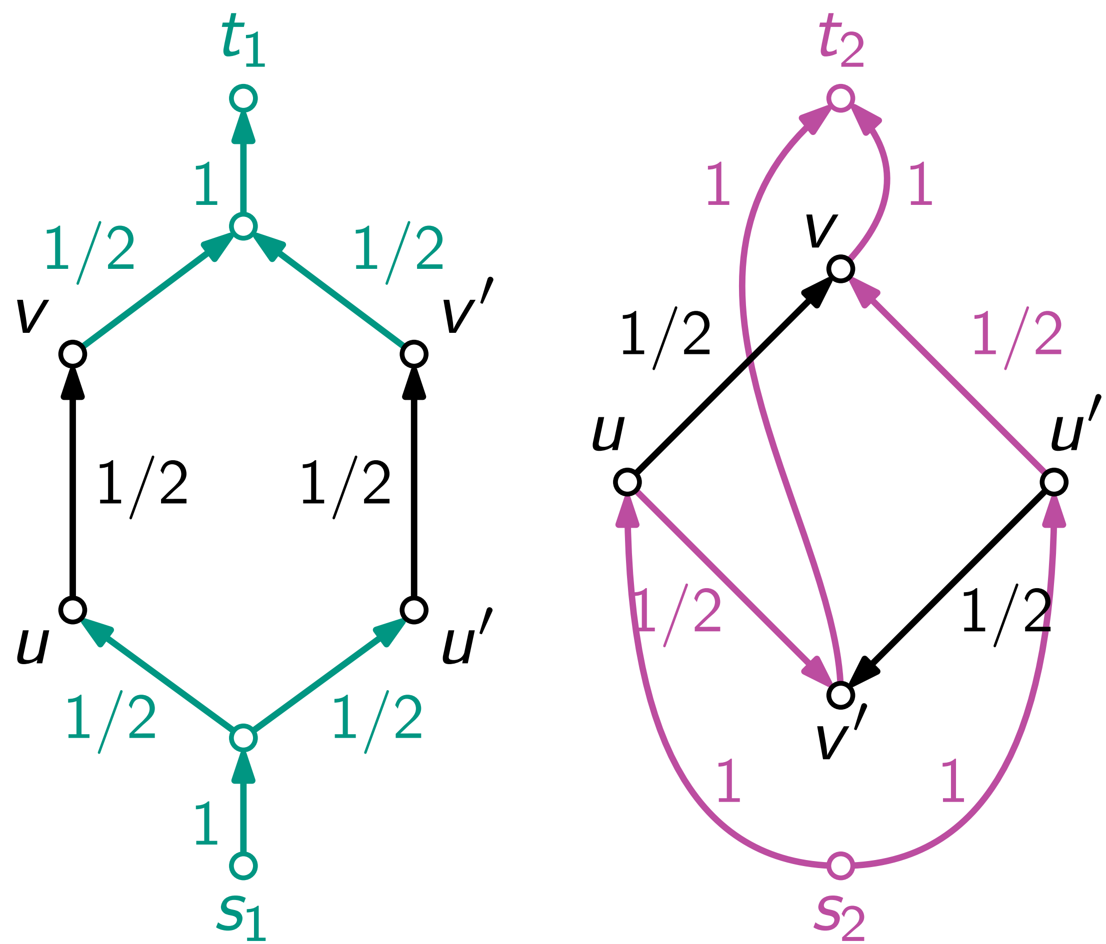

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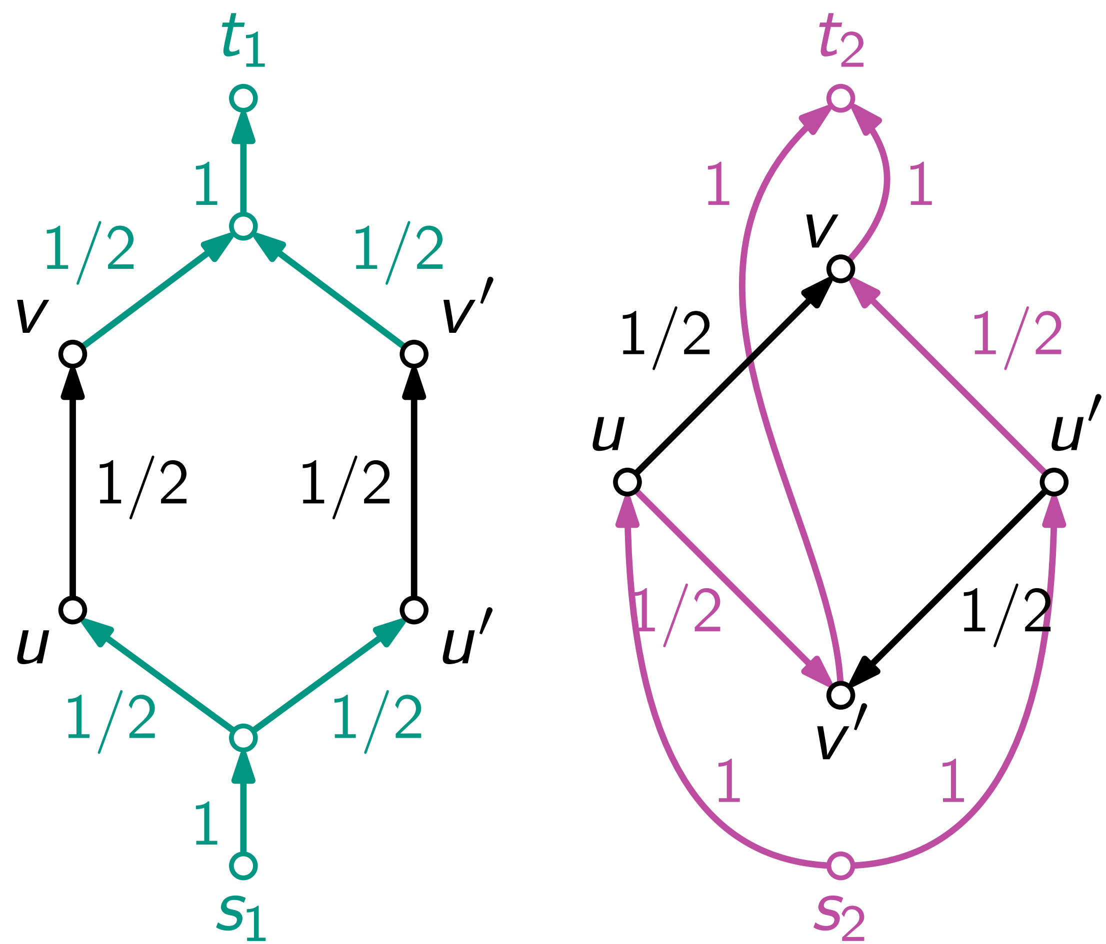
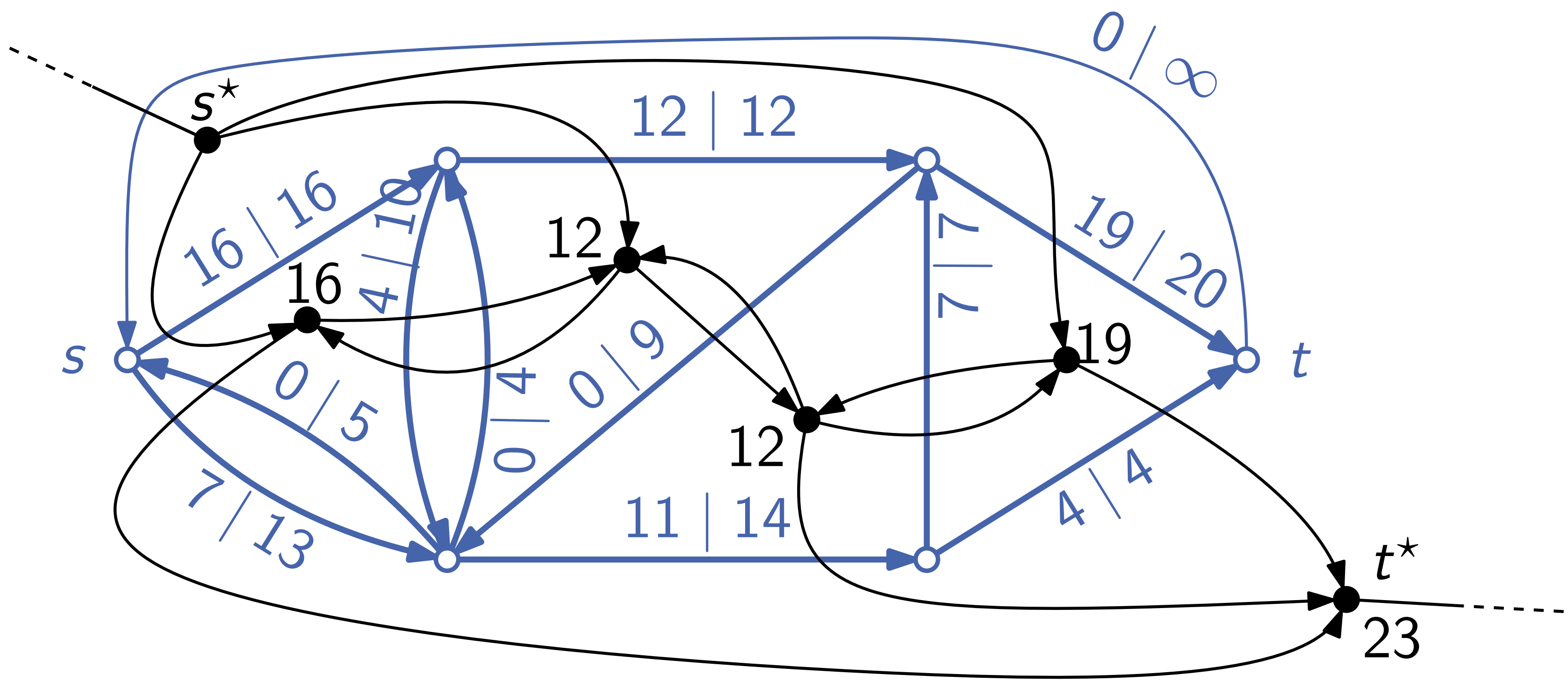


max. simultaneous real flow has values 1 and 2, but no simultaneous integer flows with these values exists

Max-Flow in Planar Graphs (w/o lower bounds)

- construct directed dual $G^\star$, set $\ell(e^\star) = c(e)$
- search for shortest $s^\star$-$t^\star$ path
- set $\varphi(u, v) = d(f_{\text{right}}) - d(f_{\text{left}})$ for $(u, v)^\star = (f_{\text{left}}, f_{\text{right}})$



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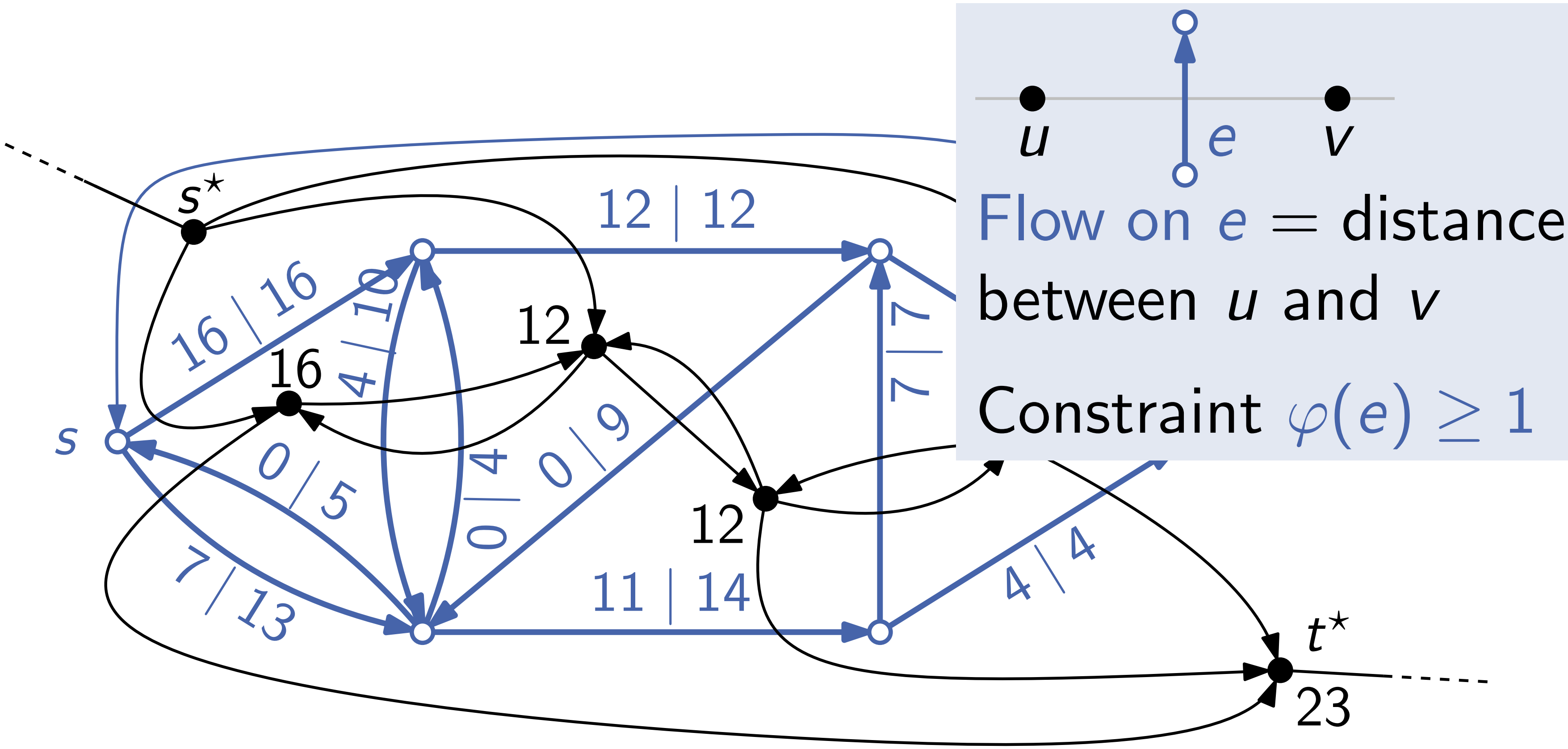

Flow on e = distance between u and v

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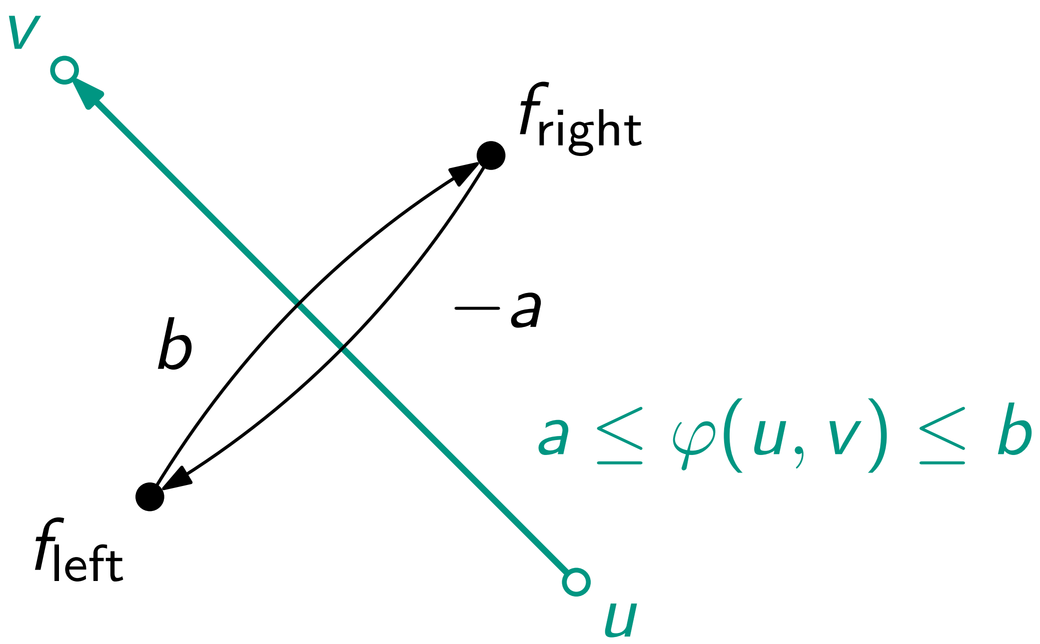
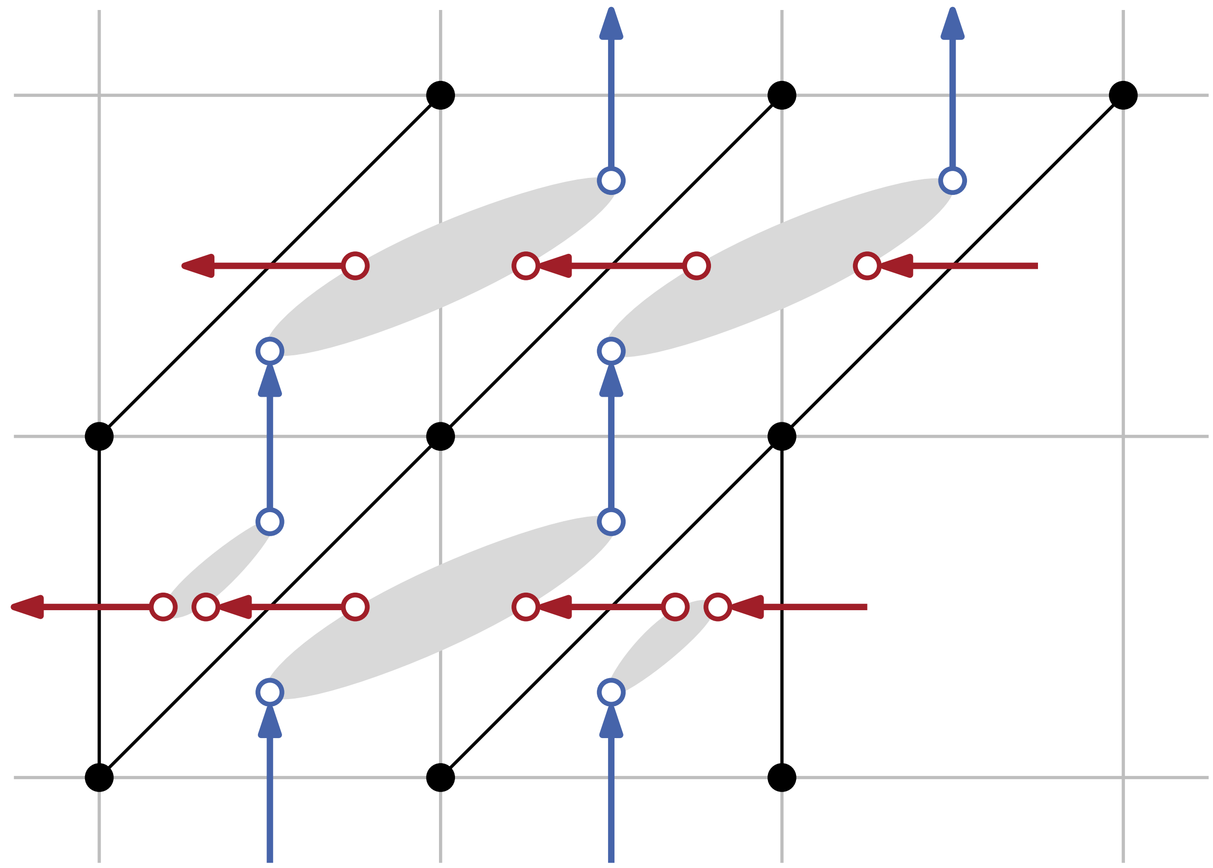
Max-Flow in Planar Graphs (w/ lower bounds)

lower bounds on the flow:

- definition:
  $\varphi(u, v) = d(f_{\text{right}}) - d(f_{\text{left}})$
- $d(f_{\text{right}}) \leq d(f_{\text{left}}) + b$
  $\Rightarrow \varphi(u, v) \leq b$
- $d(f_{\text{left}}) \leq d(f_{\text{right}}) - a$
  $\Rightarrow \varphi(u, v) \geq a$



Max-Flow and Shortest Paths



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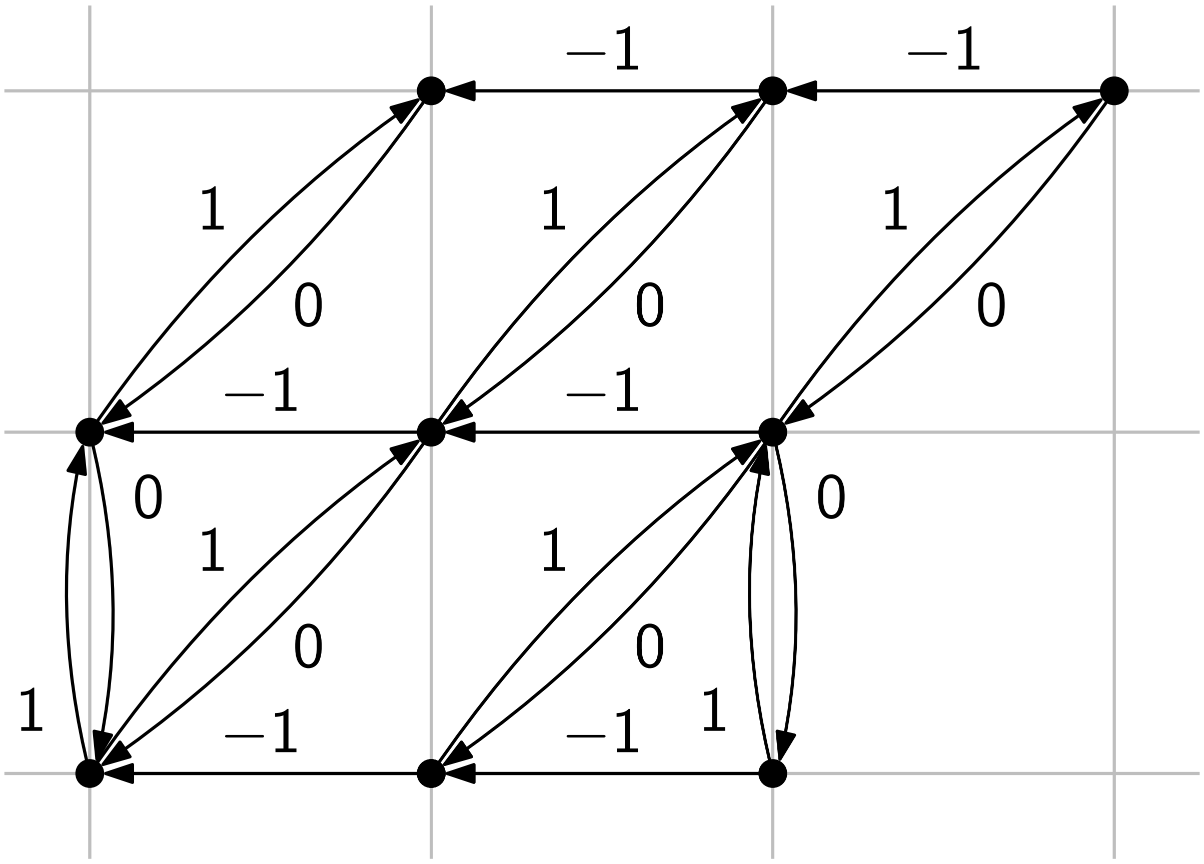
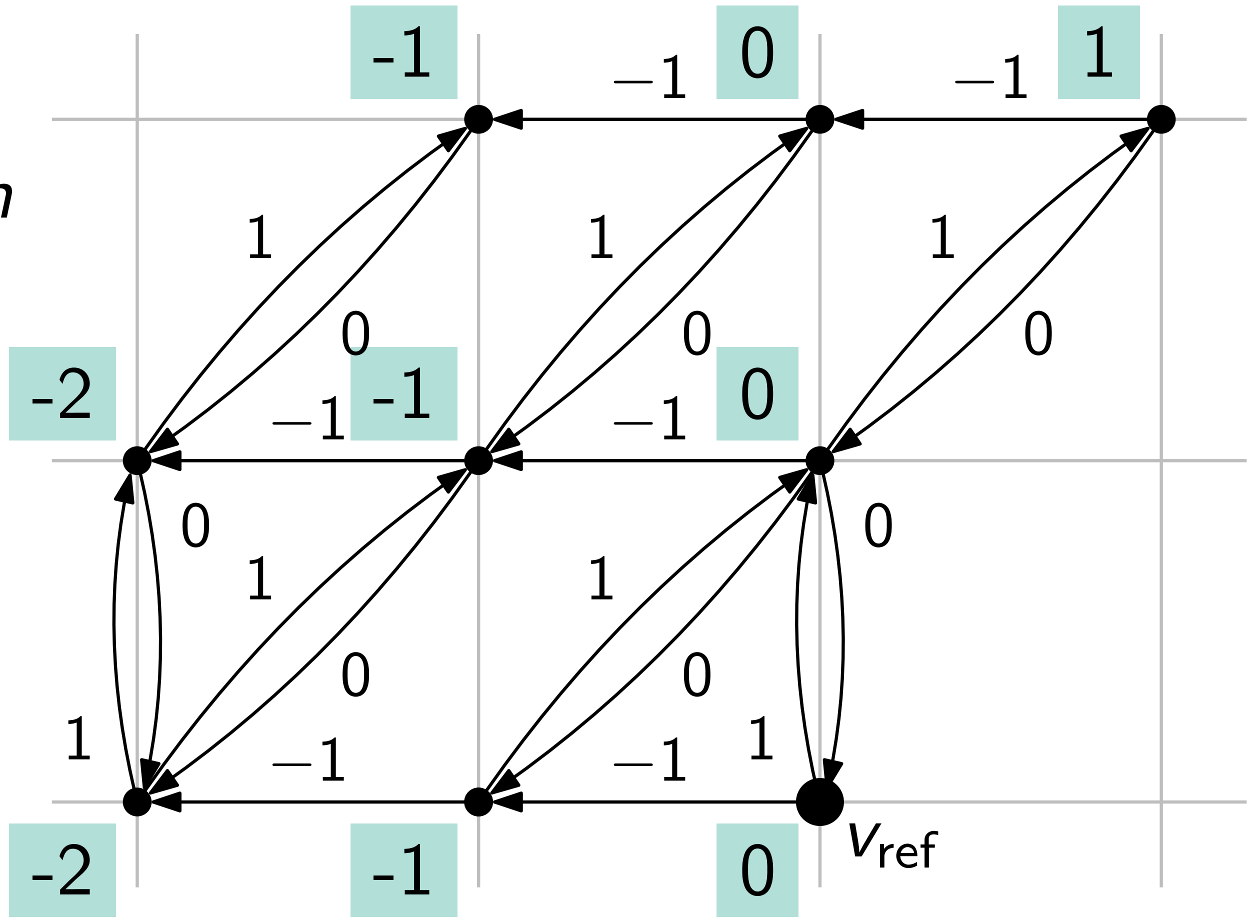


Max-Flow and Shortest Paths

- Drawing
  $O(n \log^2 n / \log \log n$



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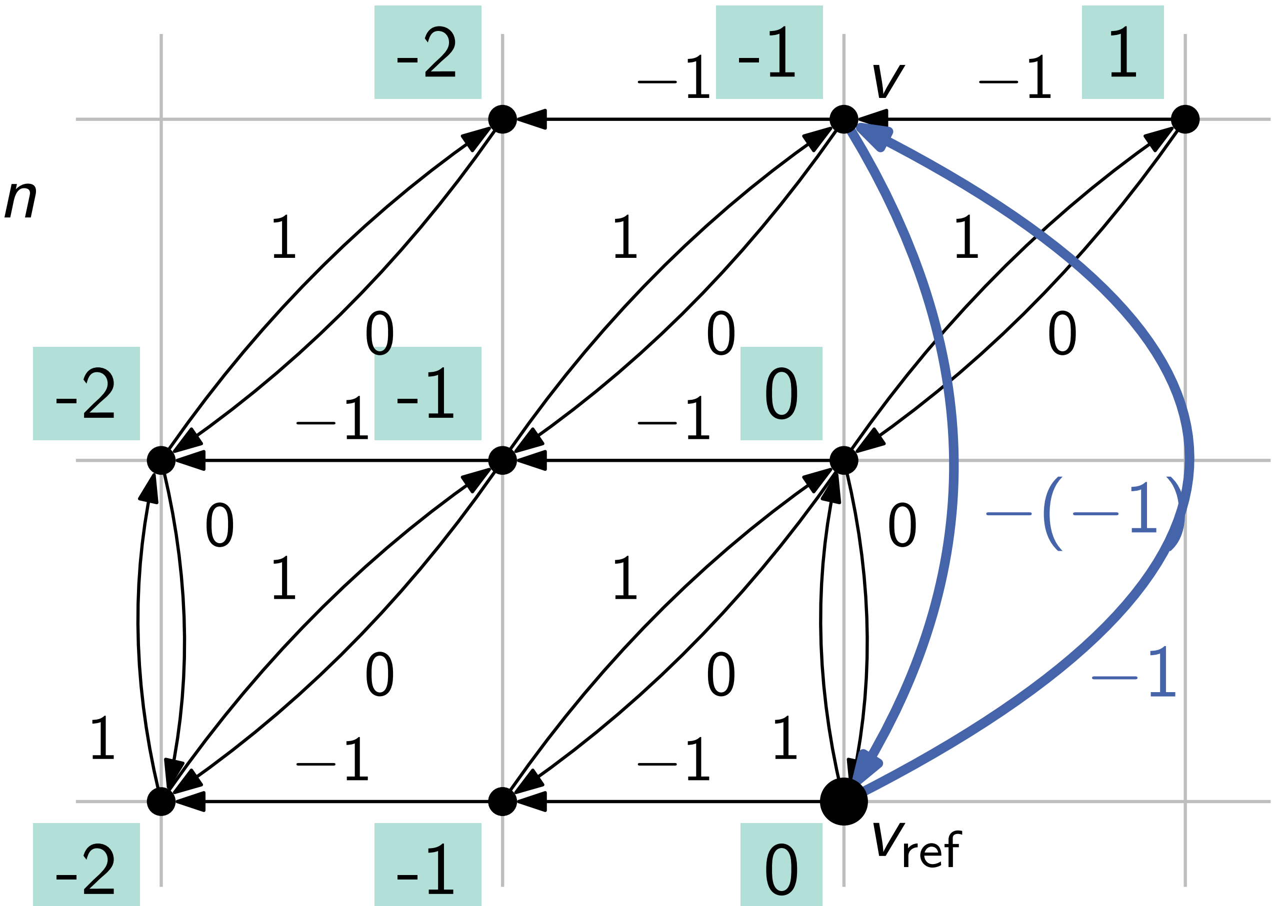

$$(v_{\text{ref}}, v) : d(v) \leq d(v_{\text{ref}}) - 1 \Rightarrow d(v) \leq -1$$

$$(v, v_{\text{ref}}) : d(v_{\text{ref}}) \leq d(v) - (-1) \Rightarrow d(v) \geq -1$$

Max-Flow and Shortest Paths

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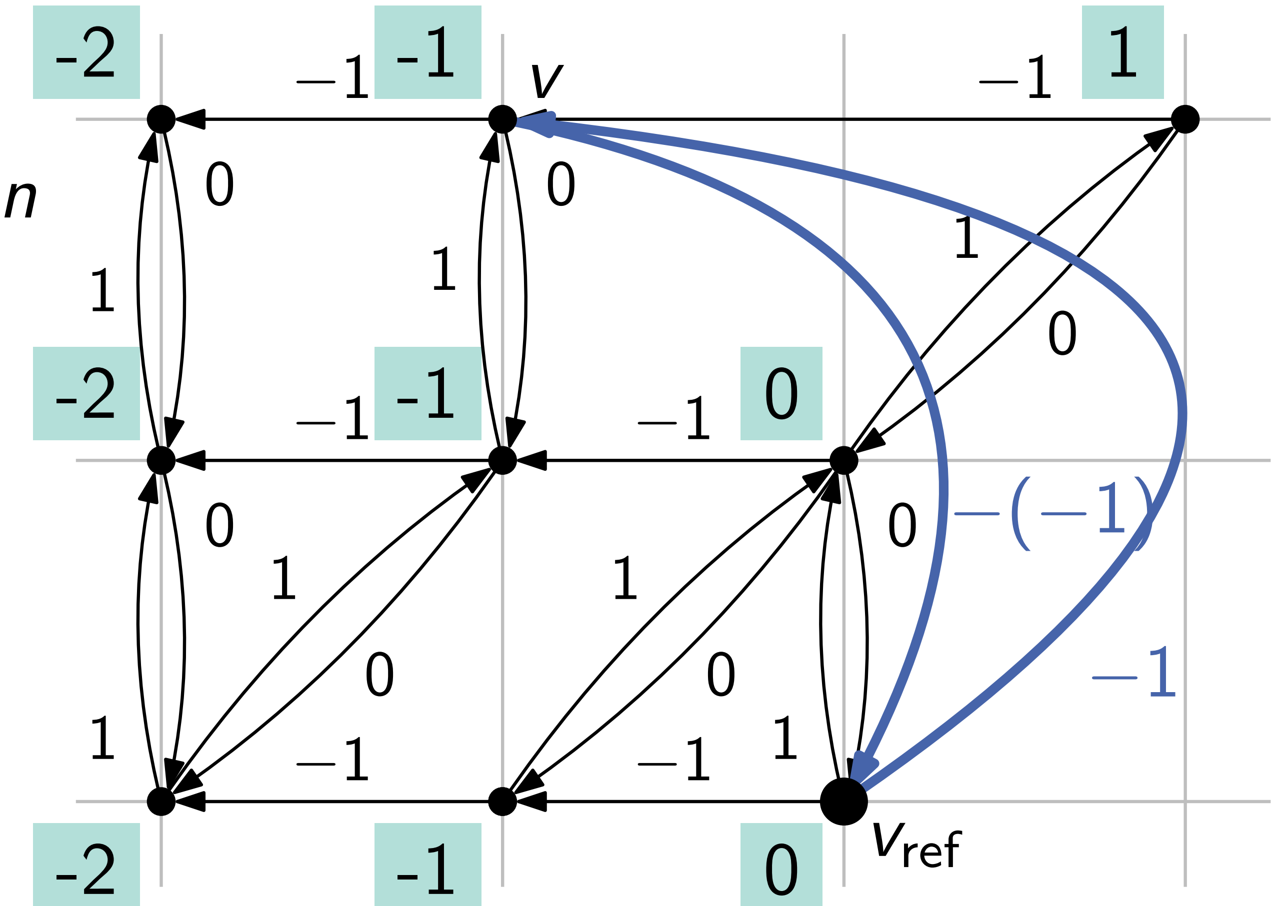

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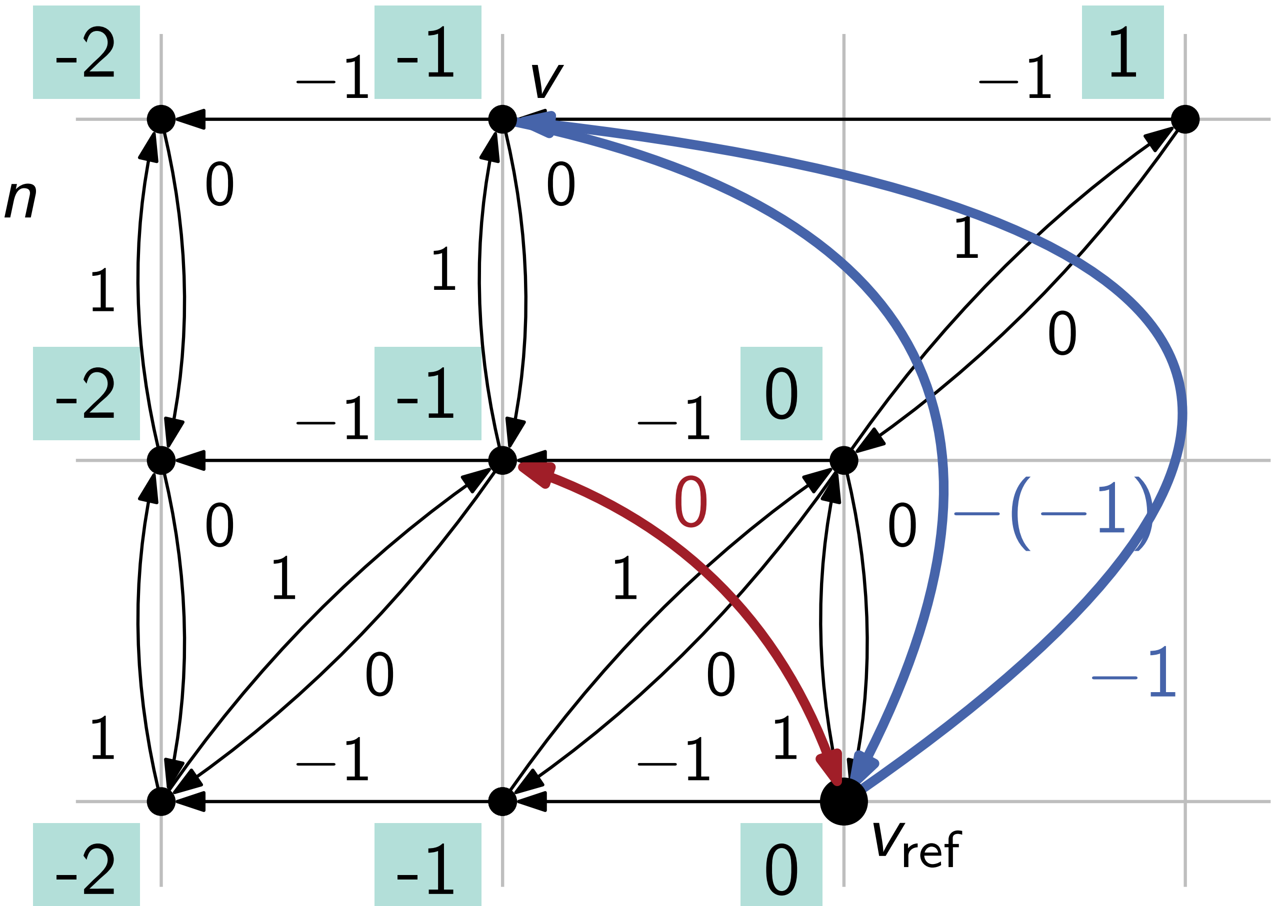
Max-Flow and Shortest Paths

- Drawing
  $O(n \log^2 n / \log \log n)$
- partial drawing extension
  $O(n^{4/3} \log n)$



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Max-Flow and Shortest Paths

- Drawing
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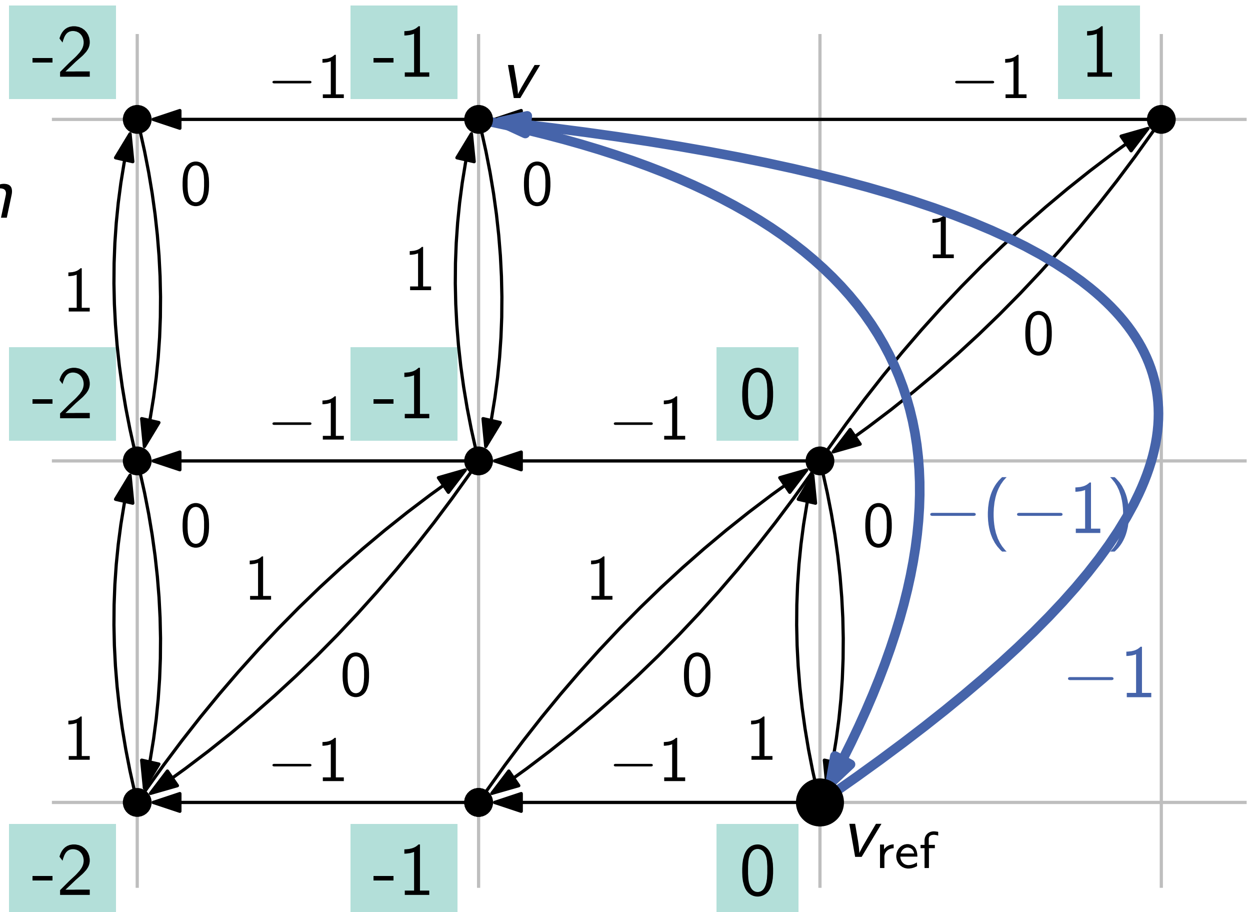
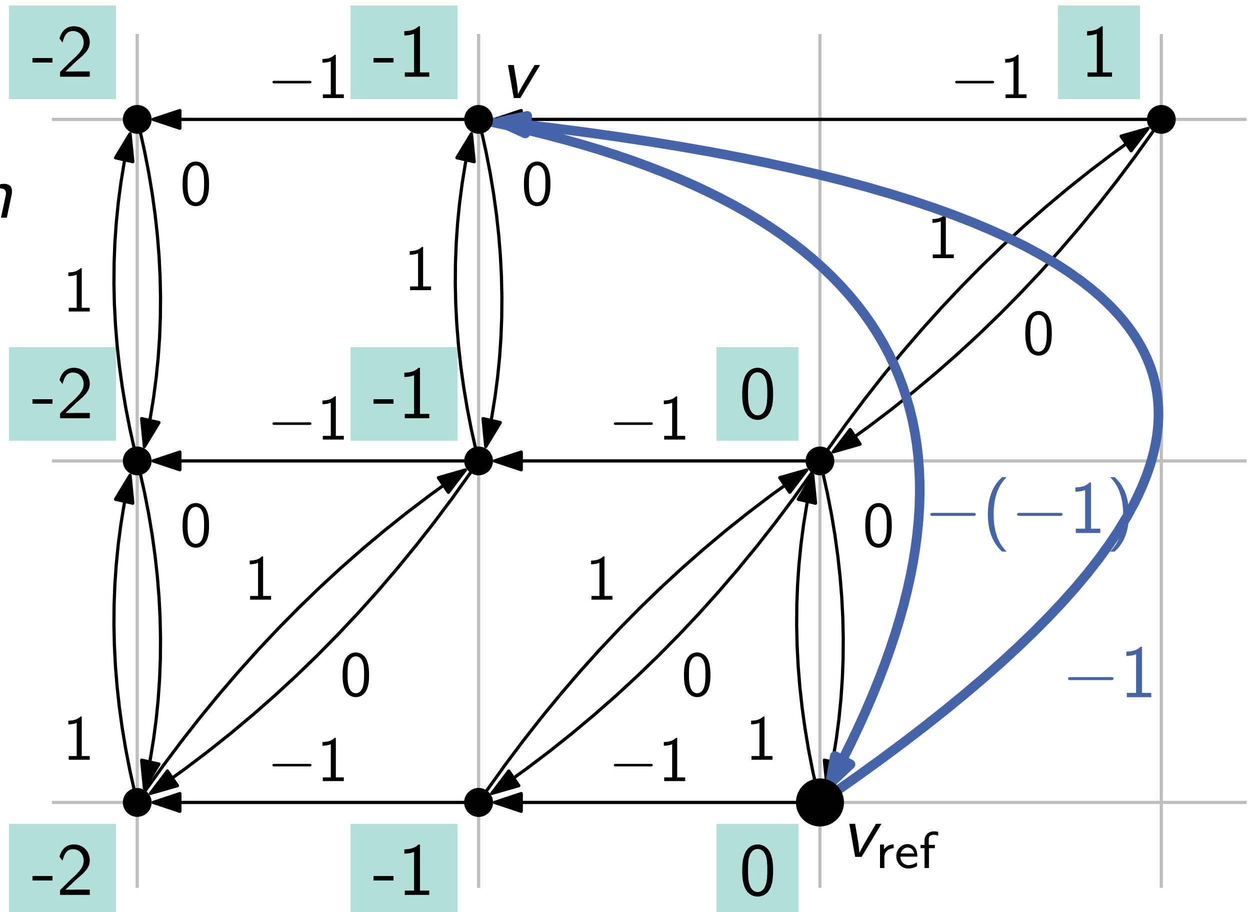


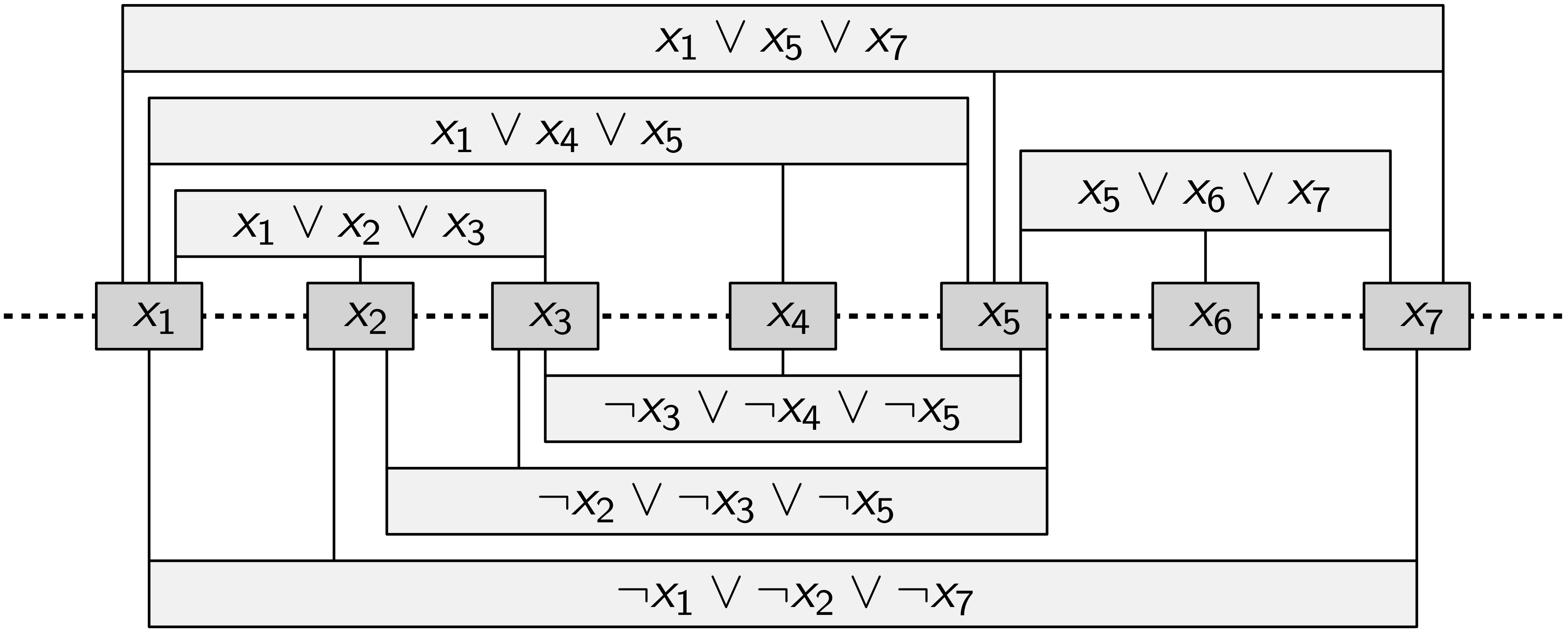
the generated drawings are *rightmost*
$d_1(v) < d_2(v) \Rightarrow$ add constraint
$d_2(v) \leq d_1(v)$ to G_2

Max-Flow and Shortest Paths

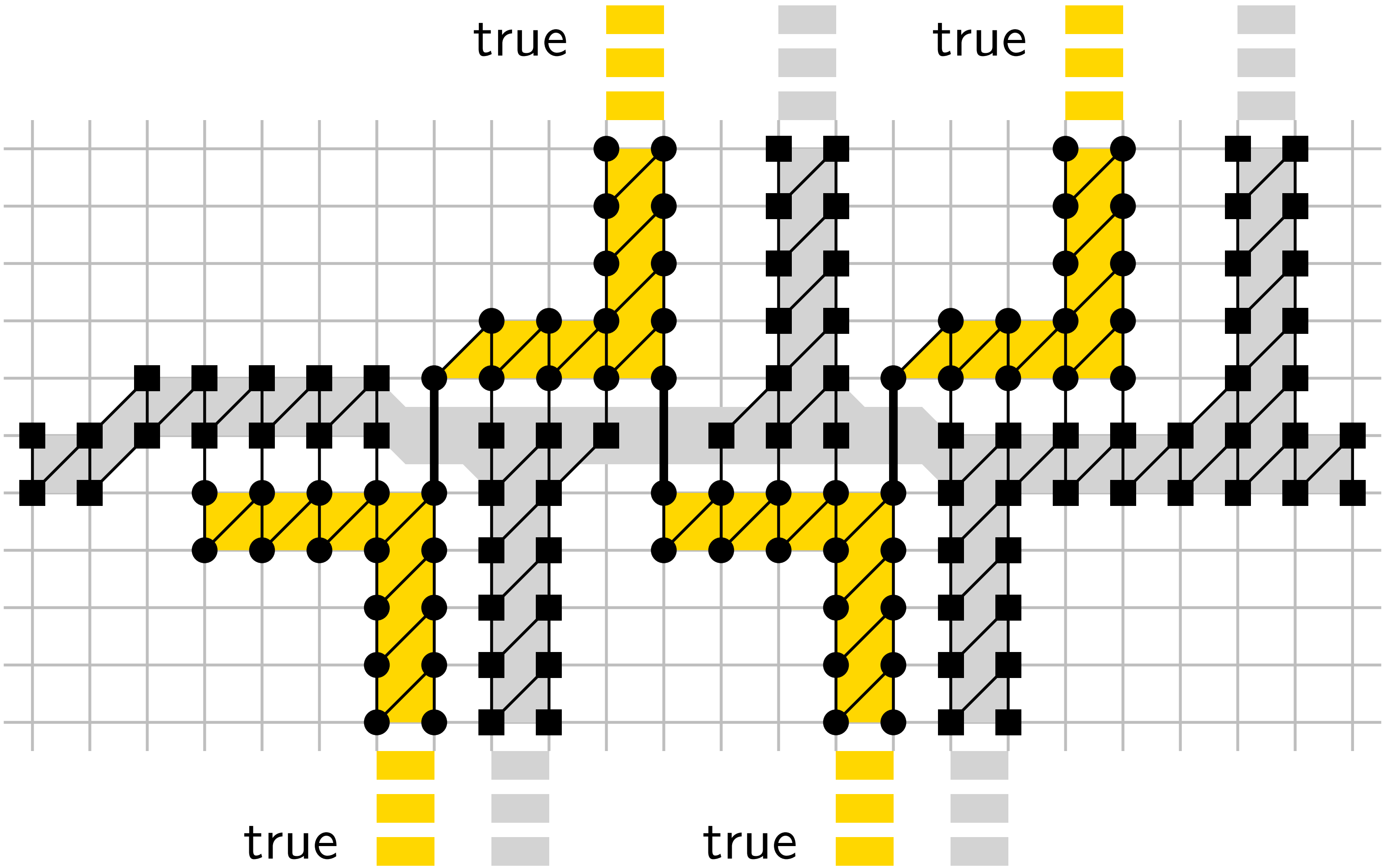
- Drawing $O(n \log^2 n / \log\log n$
- partial drawing extension $O(n^{4/3} \log n)$
- simultaneous drawings $O(n^{10/3} \log n)$
- works for $\lambda \in \mathbb{N}$
- NP-complete for "short long" edges, i.e., $\ell(v) - \ell(u) \le 2$



Rectilinear Planar Monotone 3-SAT



Variable Gadget



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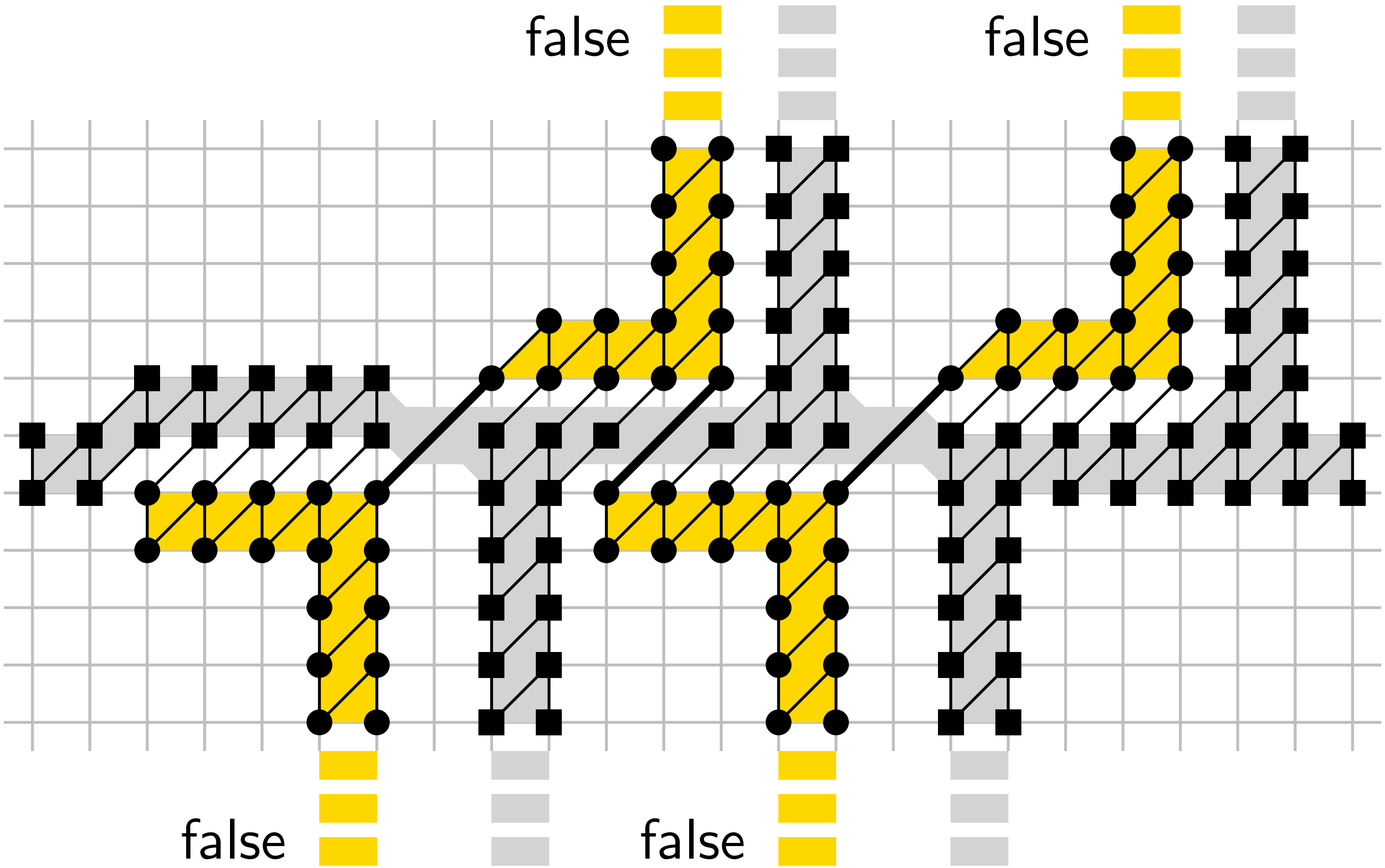
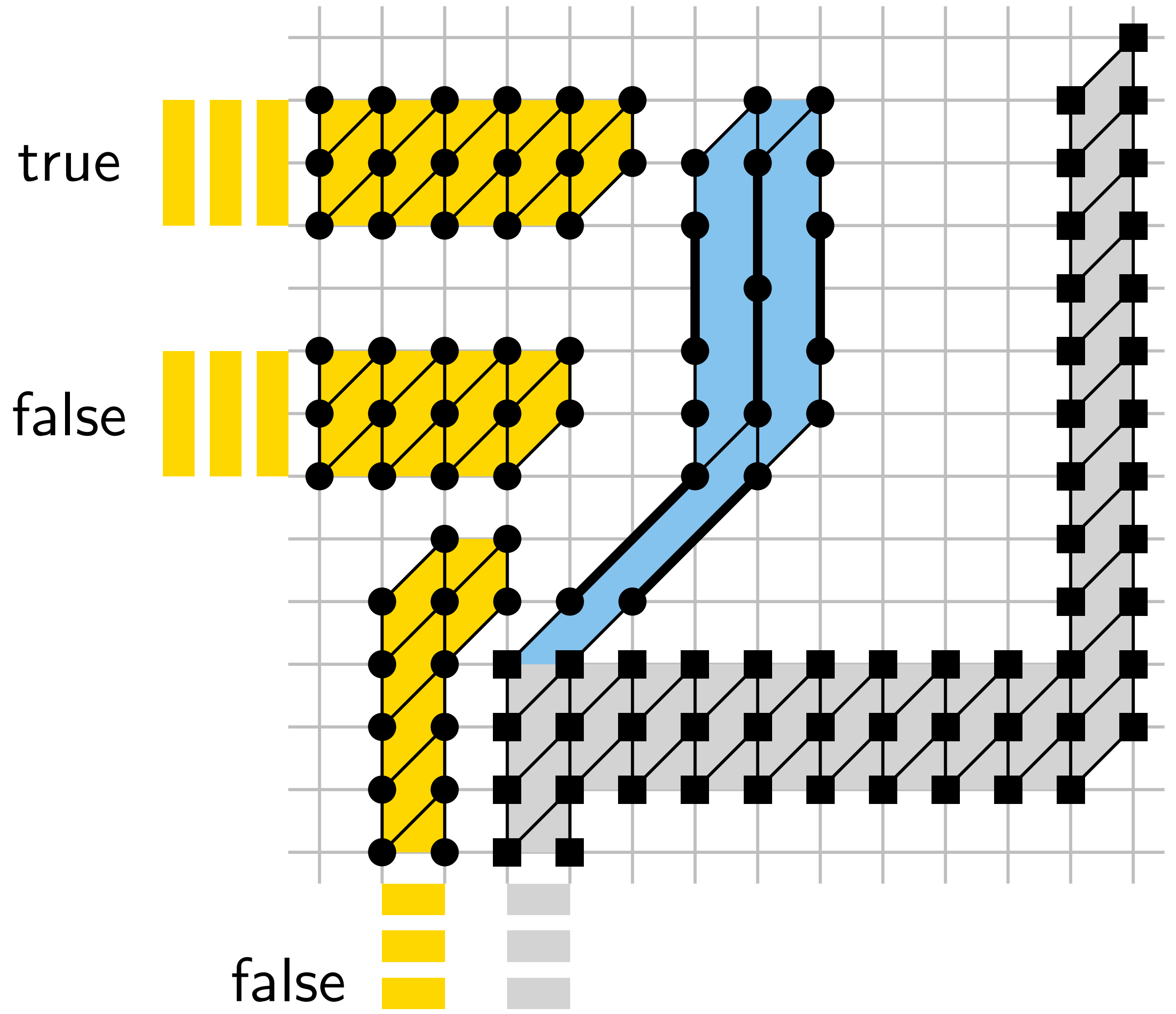


(Positive) Clause Gadget



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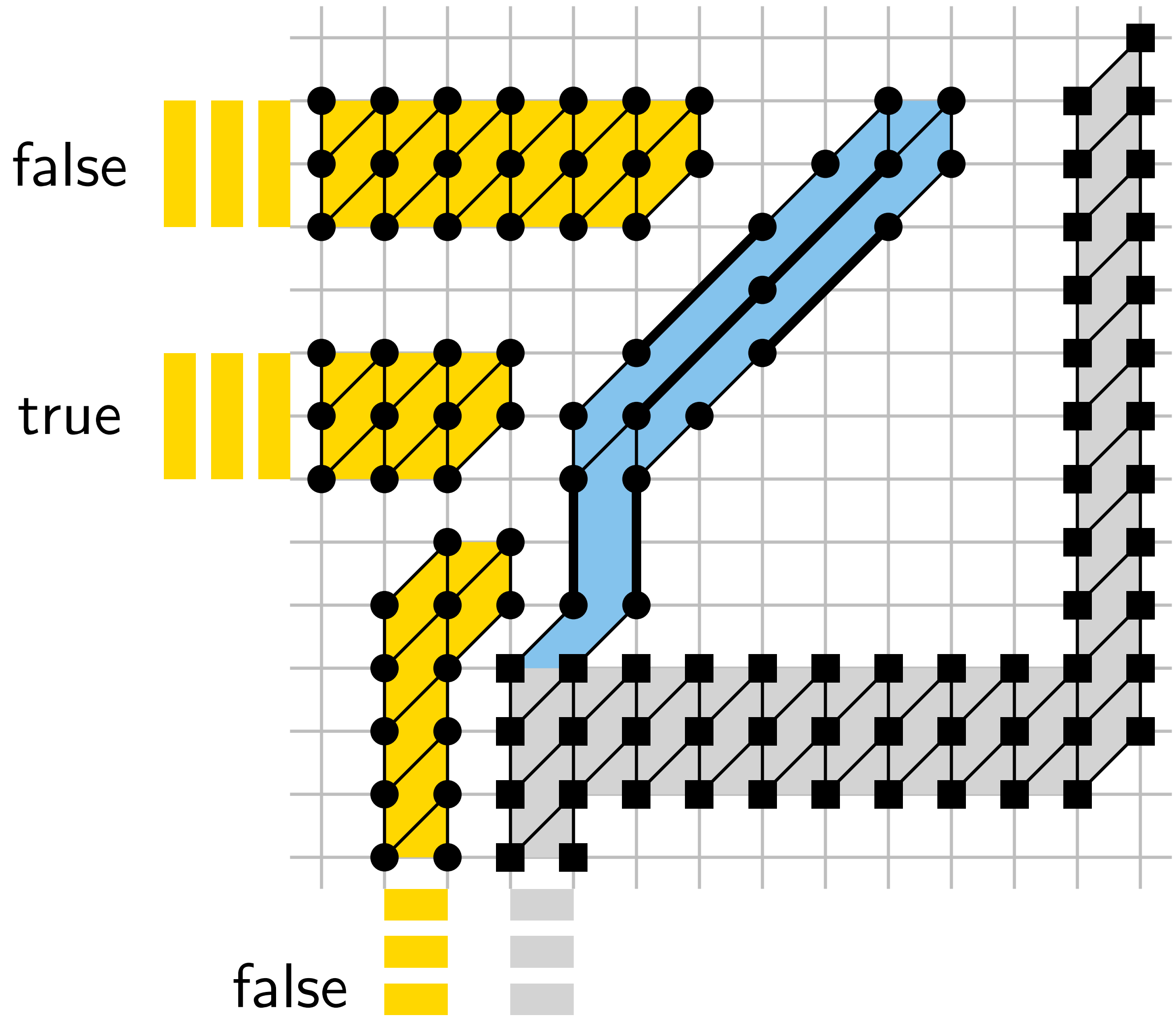

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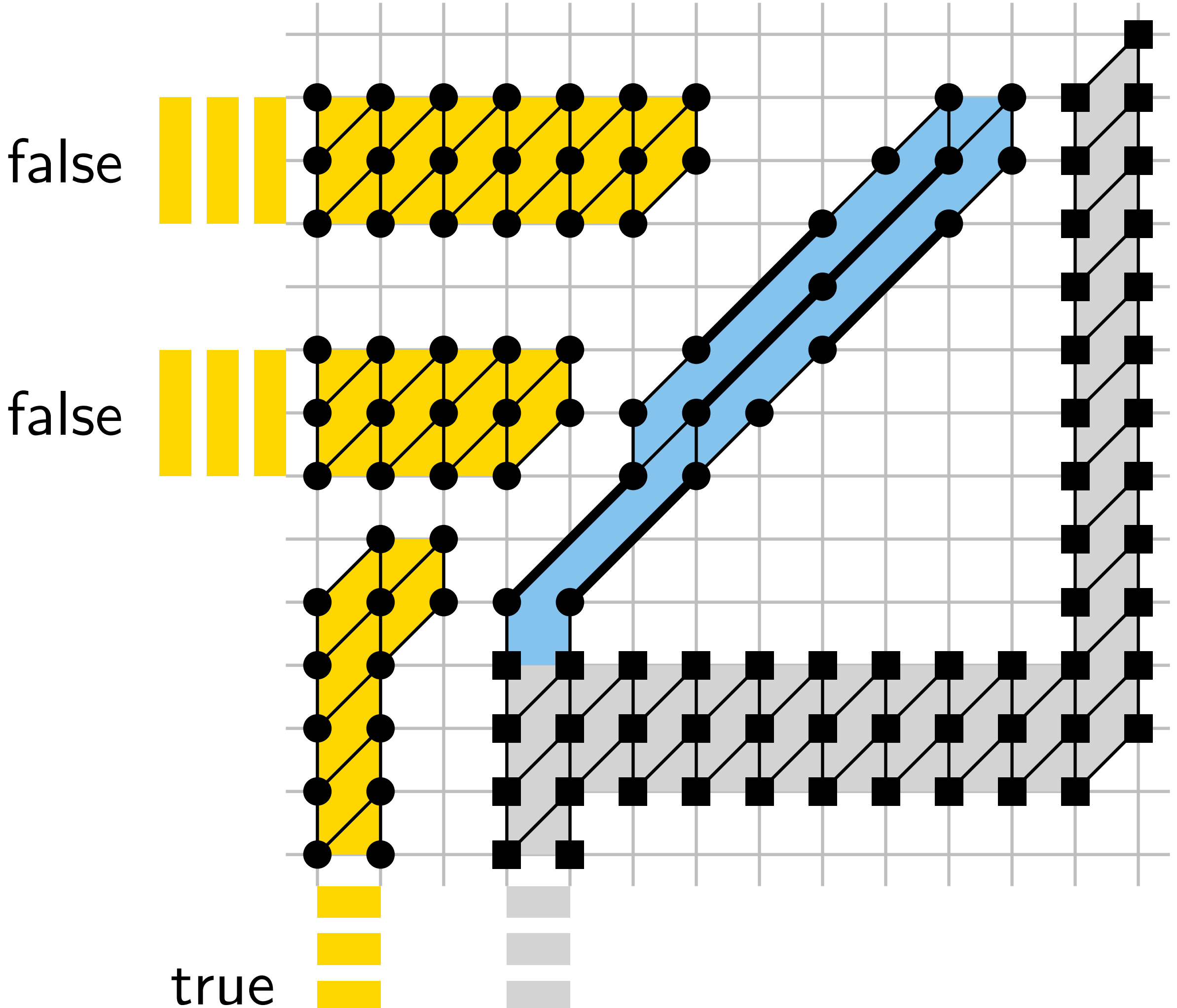

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