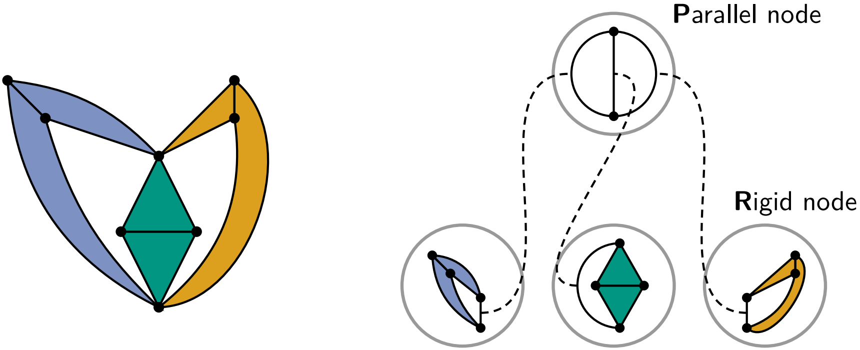
# An SPQR-Tree-Like Embedding Representation for Upward Planarity

**Guido Brückner[1], Markus Himmel[1], Ignaz Rutter[2]**

[1] Karlsruhe Institute of Technology · [2] University of Passau

# SPQR-Trees [Di Battista, Tamassia '90]

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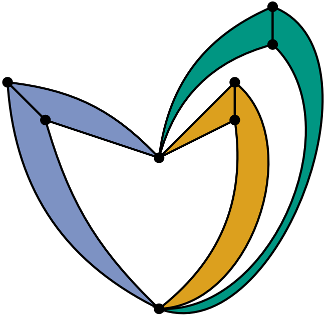

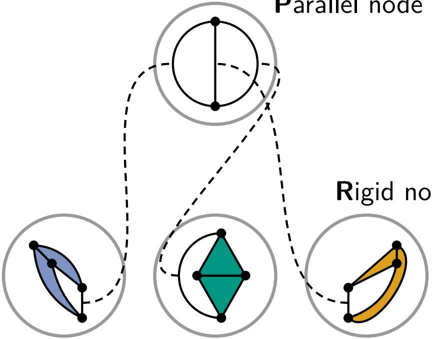


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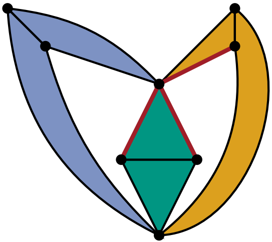

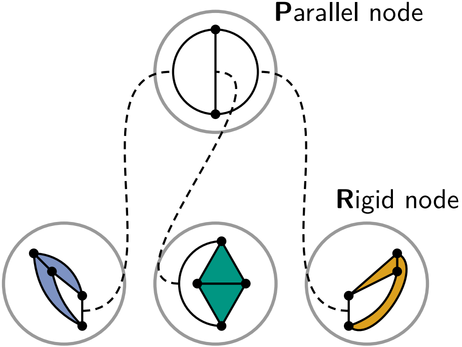


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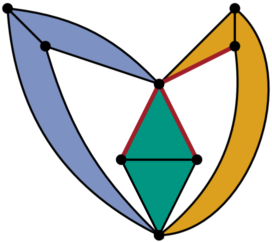

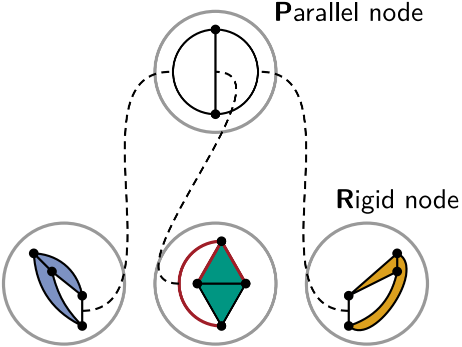


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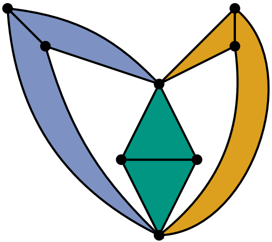

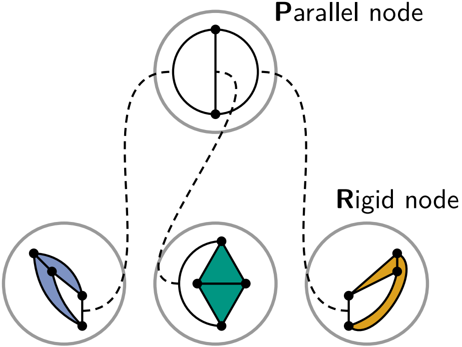


- Compactly represent all planar embeddings of a biconnected planar graph
- Have been used to efficiently solve a range of problems
  - Optimization [e.g. Didimo et al. '18], extension [Angelini et al. '15], . . .
- **Can we find a similar data structure for the upward planar case?**

# Upward planarity testing

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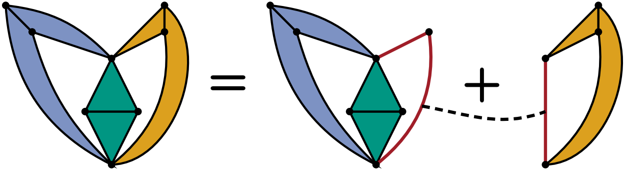
- Sufficient to only consider biconnected graphs
- Basic idea: Decomposition at 2-vertex cuts
- "Shape" of the rest of the graph $\longleftrightarrow$ suitable marker graph



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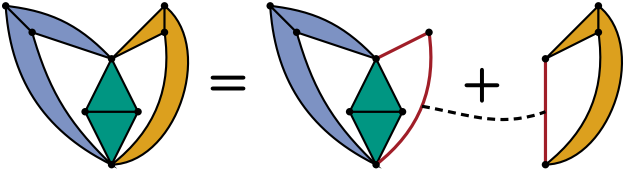
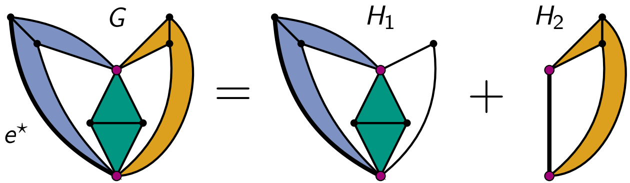


- Linear-time algorithm by Bertolazzi et al. '98 based on SPQR-trees
- Simpler algorithm by Hutton and Lubiw '96 using general decompositions

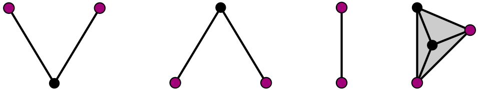
# A decomposition result by Hutton and Lubiw

Lemma



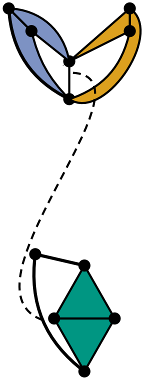

*Bijective correspondence between embeddings of $G$ and combinations of embeddings of $H_1$ and $H_2$ where*

- *Marker graphs determined by a set of rules*



- *$H_1$ or $H_2$ is single component*
- *Fixed edge $e^\star$ or its marker are leftmost*

# Decomposition Trees for Upward Planar Embeddings



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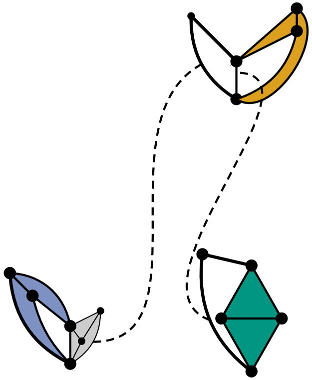

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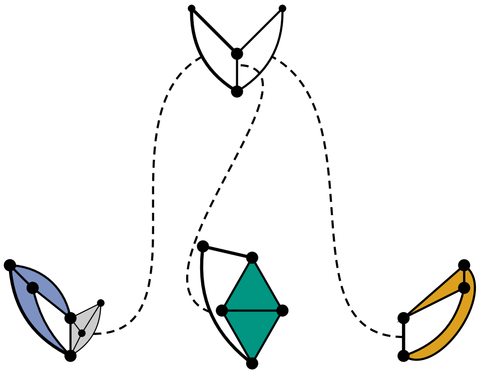

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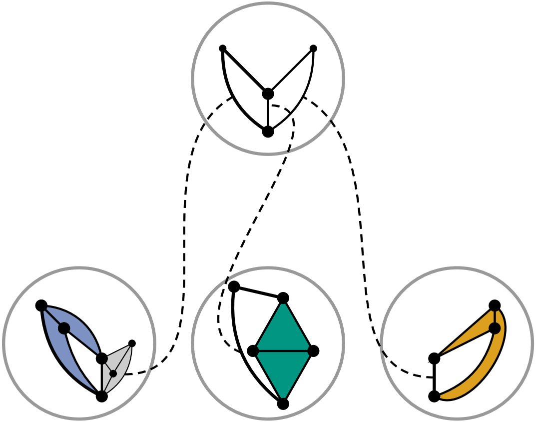

- Upward planar embedding $\longleftrightarrow$ upward planar skeleton embeddings
- Each sequence of decompositions $\rightsquigarrow$ new characterization of upward planar embeddings

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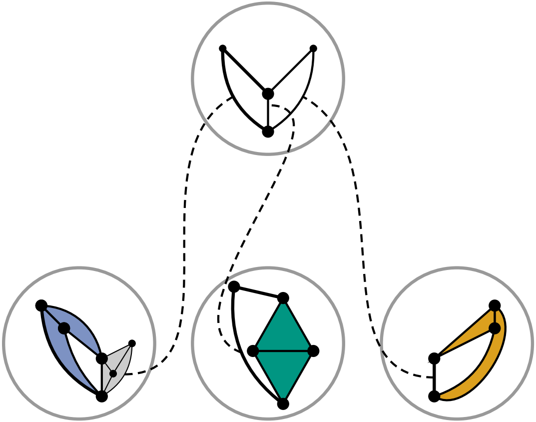

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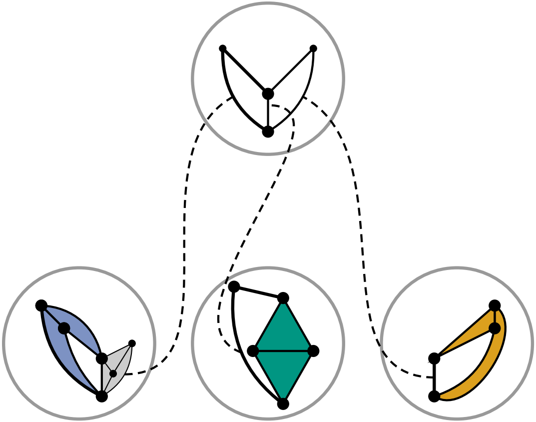
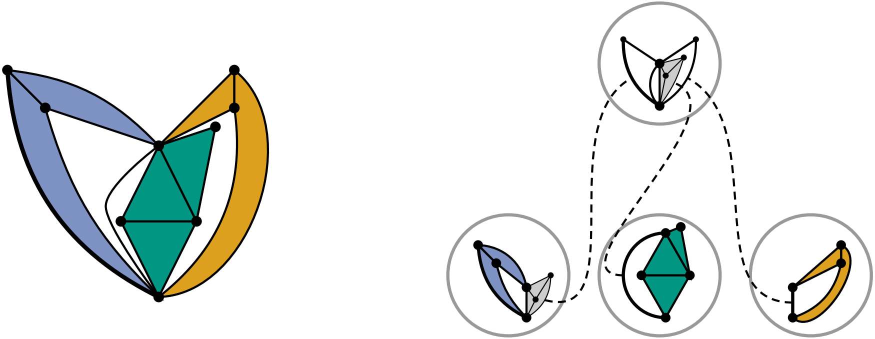
- Each sequence of decompositions $\rightsquigarrow$ new characterization of upward planar embeddings
  - Actually, order is irrelevant

# Decomposition Trees for Upward Planar Embeddings



- Important question: Which decomposition tree should we use?
- SPQR-tree is nice for planar embeddings, but offers too many choices
- Idea: Modify SPQR-tree to have upward planar skeletons

# P-Node Splits



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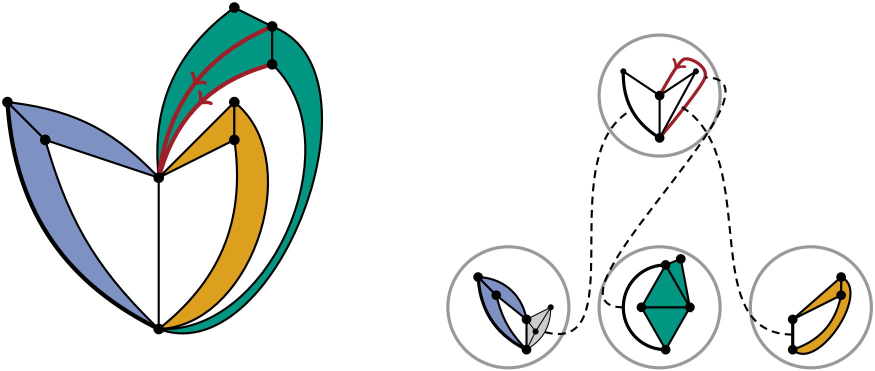

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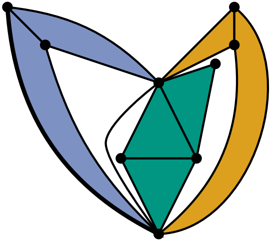
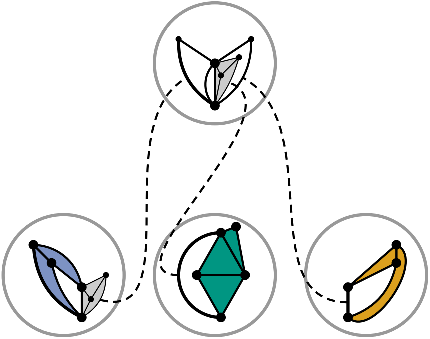
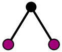
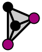




- Problem: Permuting edges at P-nodes ⇝ non-upward-planar skeleton
- Solution: Split P-nodes by marker type
  - Relevant here: ,  and .

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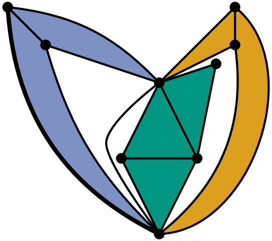

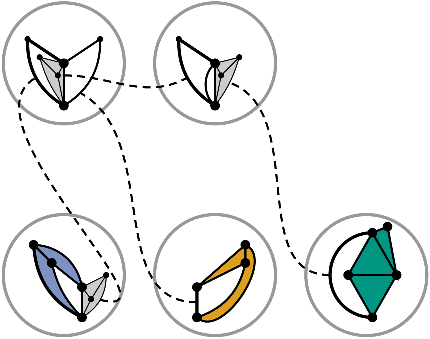

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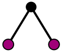
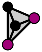
  - Relevant here: ,  and .

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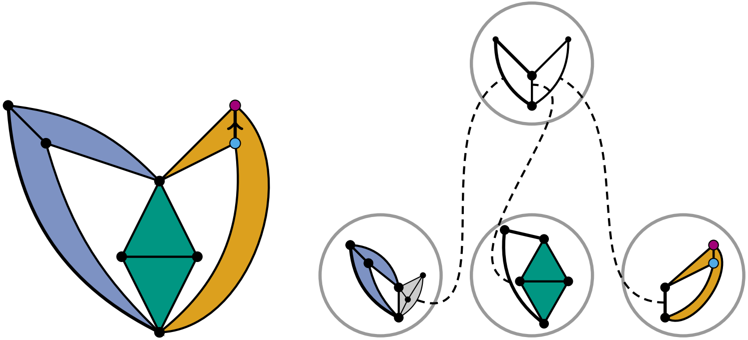

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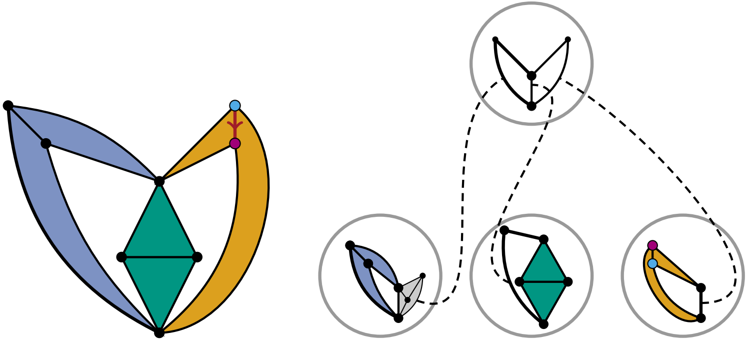

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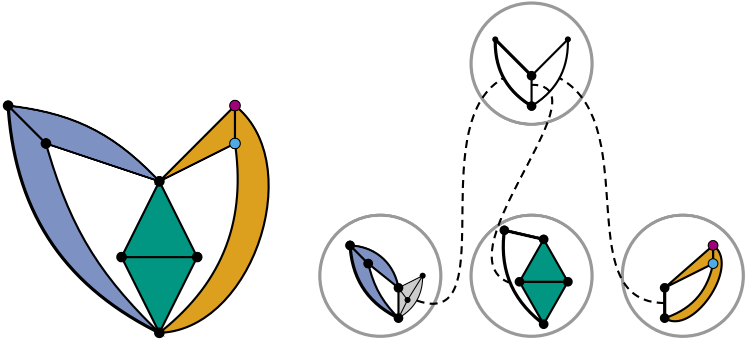

- Problem: Flipping operation at R-nodes $\rightsquigarrow$ non-upward-planar skeleton
- Solution: Contract arcs of the tree that do not give embedding choices
  - Upward planarity test for embedded single-source graphs [Bertolazzi et al. '98]

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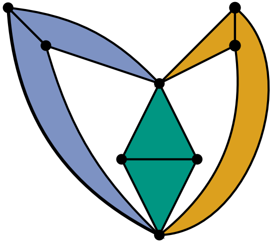

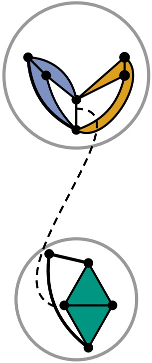

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## The UP-Tree

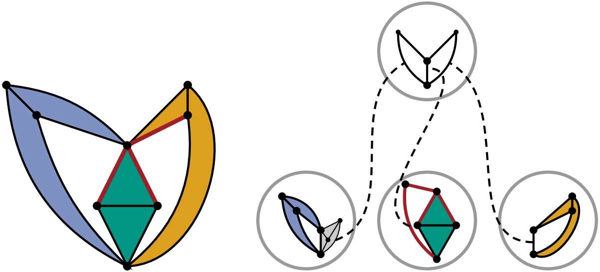
- SPQR-tree + P-node splits + arc contractions =: UP-tree

Theorem

*For each biconnected single-source DAG $G$ and $e^\star$ incident to $s$ there is a decomposition tree $\mathcal{T}$ computable in linear time that*

- *represents the upward planar embeddings of $G$ in which $e^\star$ is leftmost*
- *does so using P-nodes and R-nodes*

- Example: Partial *upward* embedding problem solvable in quadratic time



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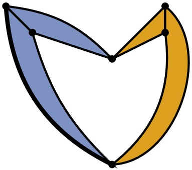
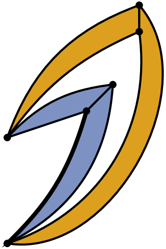
- NB: Dependency on $e^\star$ is necessary:





# Conclusion

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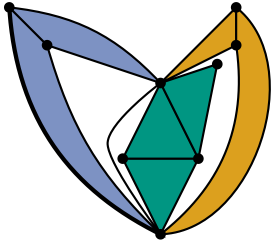

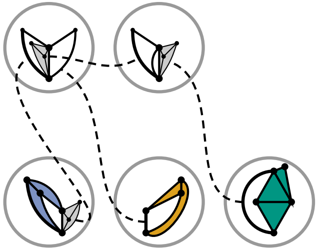

Future work: Survey more algorithms that use SPQR-trees and adapt to upward planar case