On the Edge-Vertex Ratio of Maximal Thrackles

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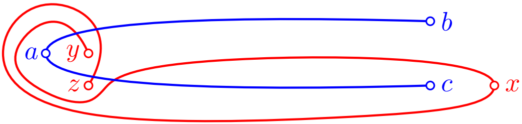
September 27, 2019

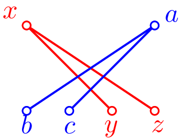
Thrackles and Conway's Conjecture

Definition ((Geometric) Thrackle)

- Topological (geometric) drawing T of a graph G
- Any two edges in T have **exactly one** point in common, either:
  - at a common endpoint, or
  - at a proper crossing.



Kynčl's Example



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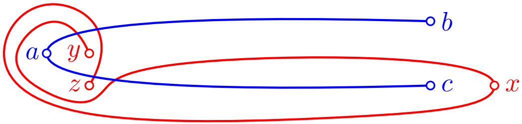

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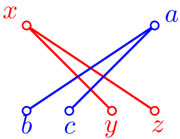

Conjecture (Conway)

Thrackles satisfy $|E(T)| \leq |V(T)|$.

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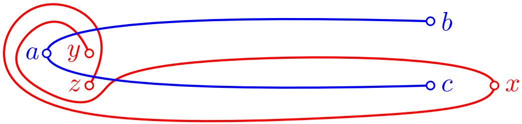

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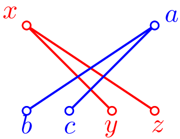

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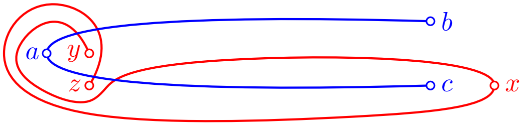

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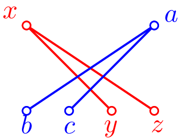

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Thrackles with edge-vertex-ratio $\varepsilon(T) = 1$ *are maximal thrackles.*

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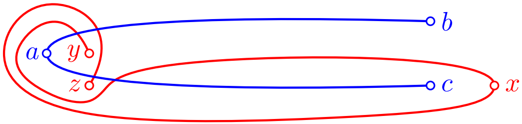
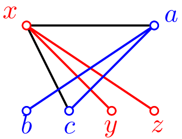


maximal



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Our Contribution

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All of these results except for the $\frac{5}{6}$ are essentially best possible.

Maximal Geometric Thrackles

Theorem

There exist

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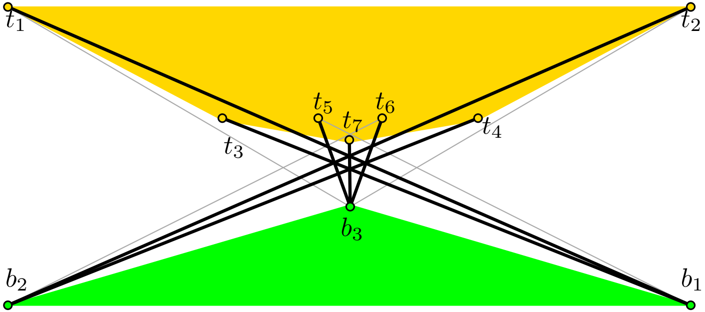

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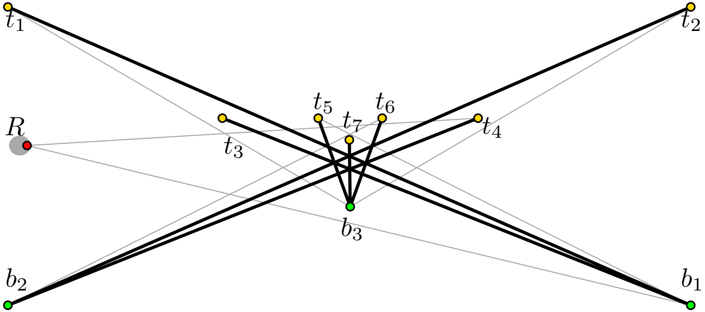

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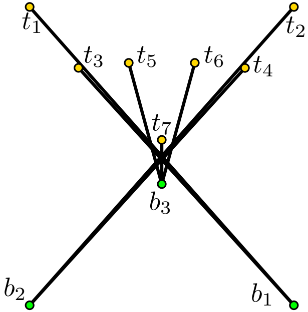

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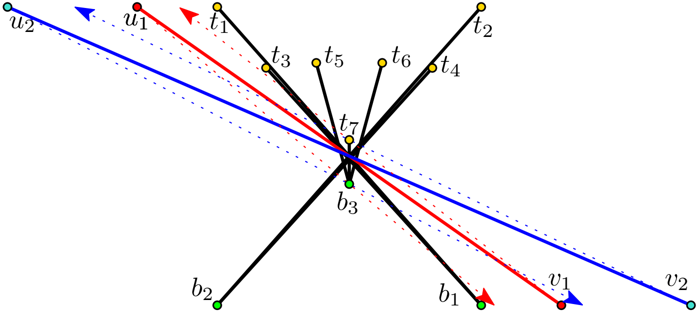

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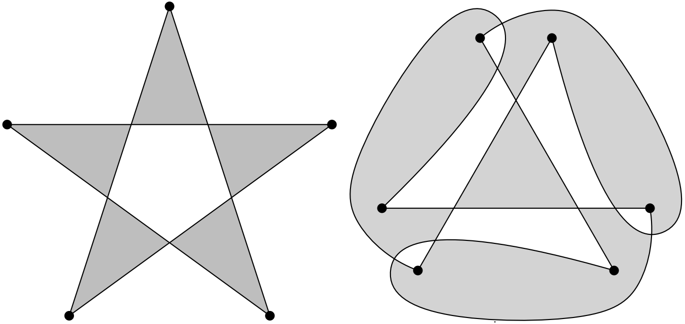

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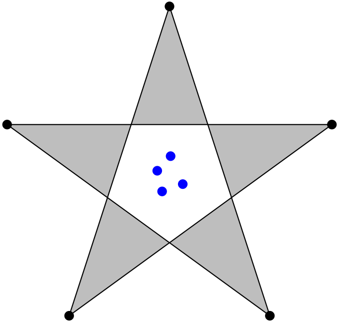

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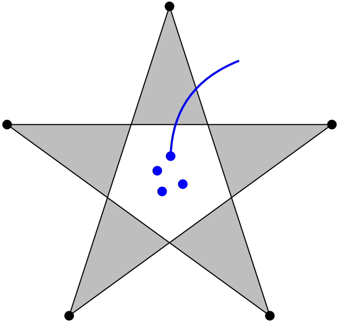

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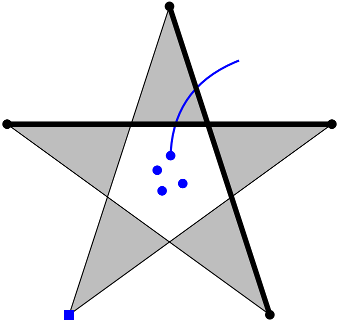

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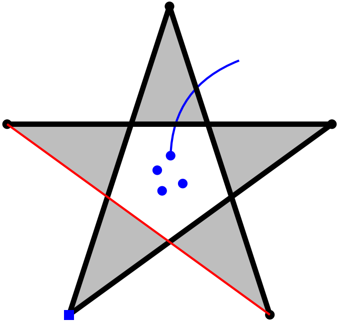

Maximal Thrackles without isolated vertices

Belt Construction 1

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There exist maximal thrackles T': $\delta(T') = 1, \varepsilon(T') = \frac{5}{6}$.

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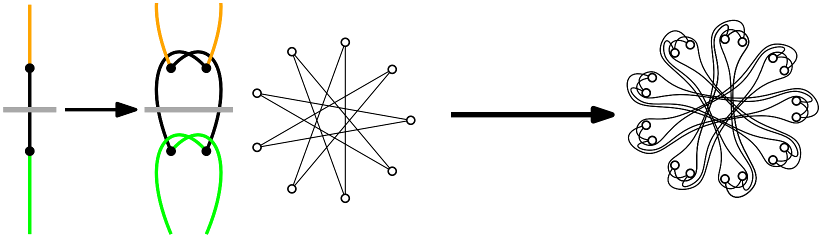
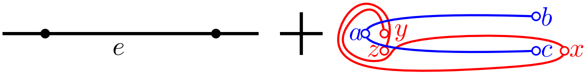
Belt construction of Woodall (1972)



Kynčl belt construction



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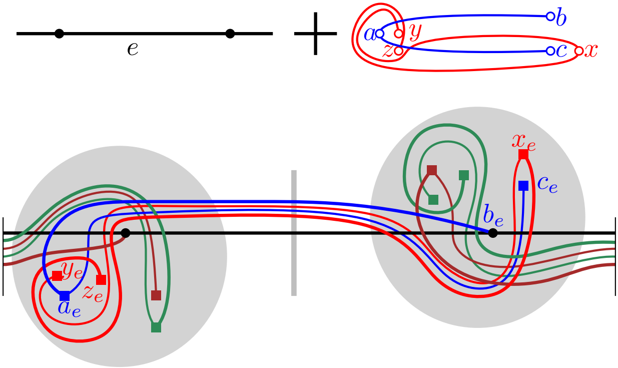

Overview+Open Problems

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- Are there any maximal matching thrackles?
- Can you prove a better lower bound than $\frac{1}{2}$?
- Are there any other better examples than Kynčl's example?
- Can you lower the constant 5 for maximal geometric thrackles without isolated vertices?
- Does Conway's Conjecture $\varepsilon(T) \leq 1$ hold?

Thank you for your attention!