Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

Stick Graphs with Length Constraints

Steven Chaplick, Philipp Kindermann, Andre Löffler, Florian Thiele, Alexander Wolff, Alexander Zaft, and **Johannes Zink**

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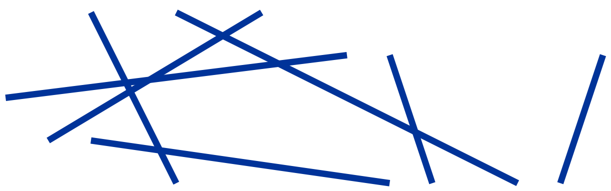

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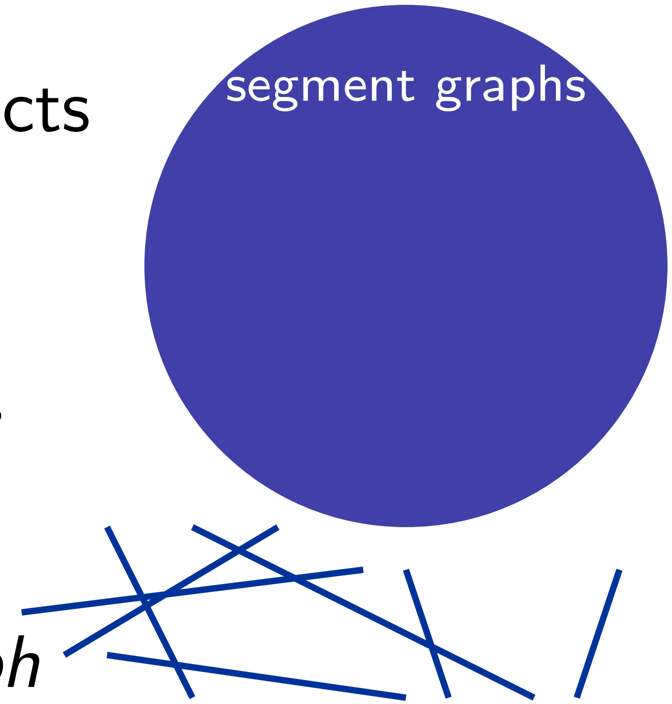

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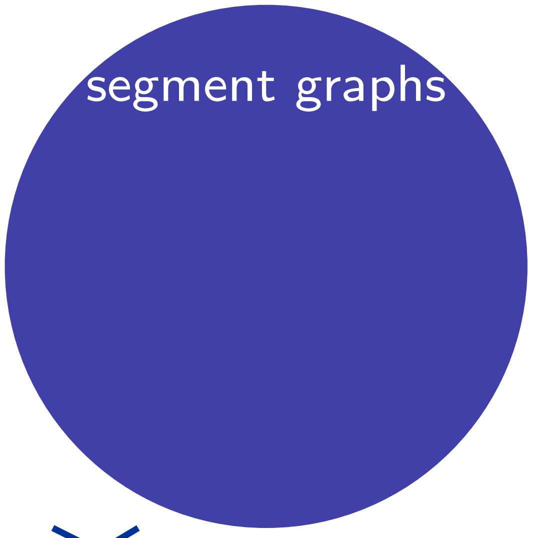

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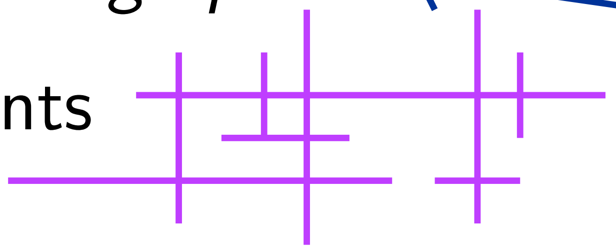

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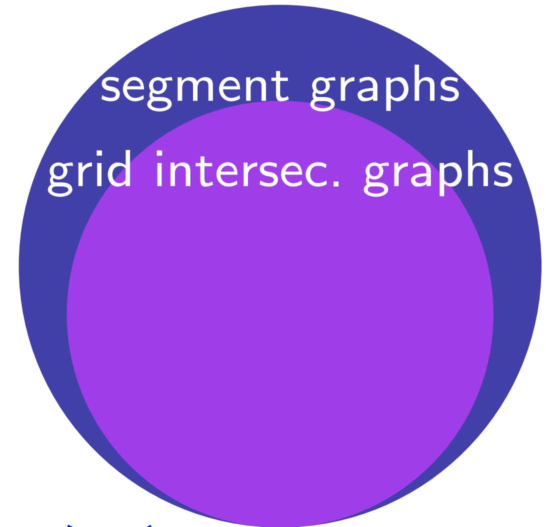

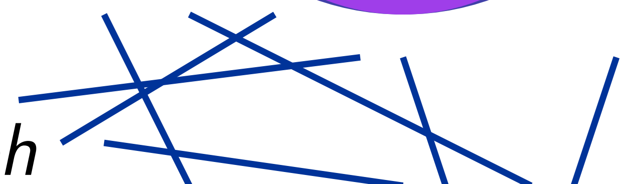

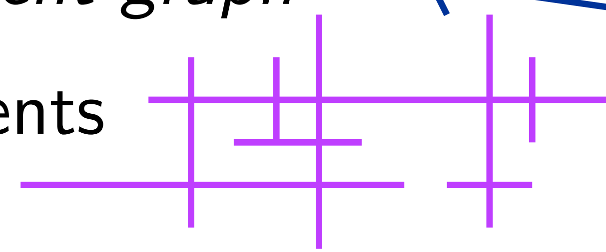

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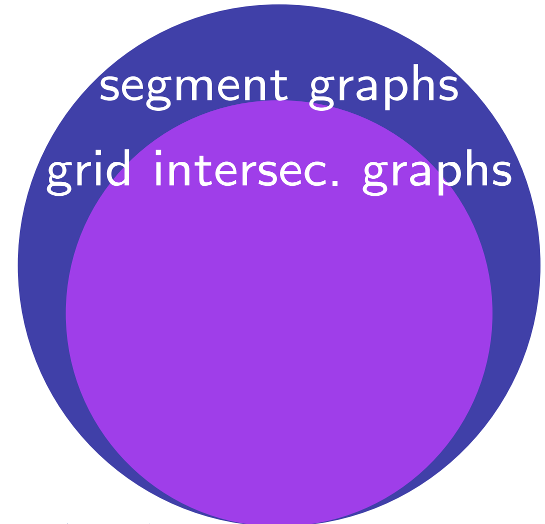

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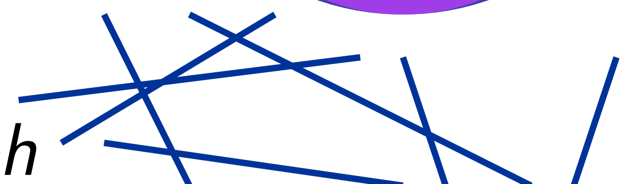

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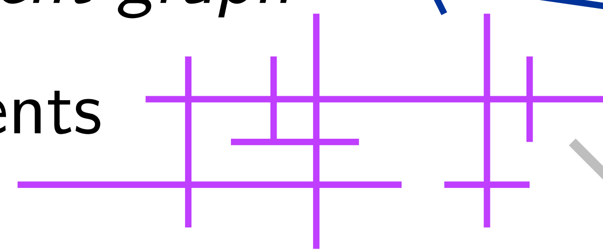

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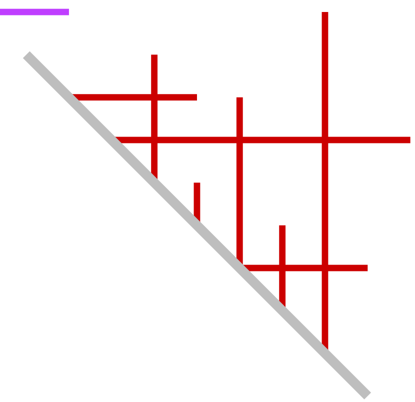

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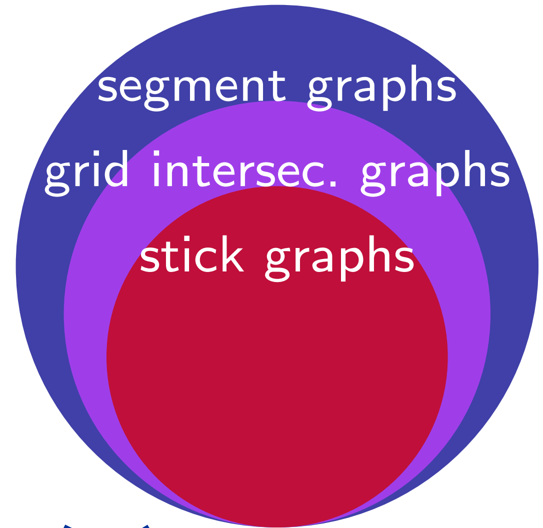

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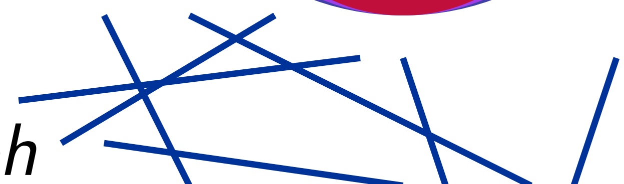

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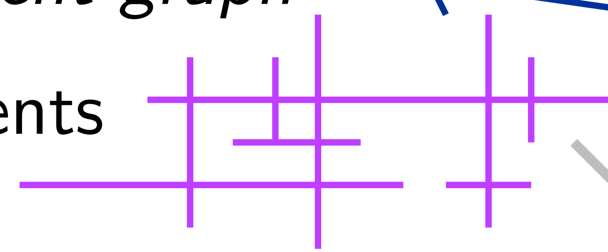

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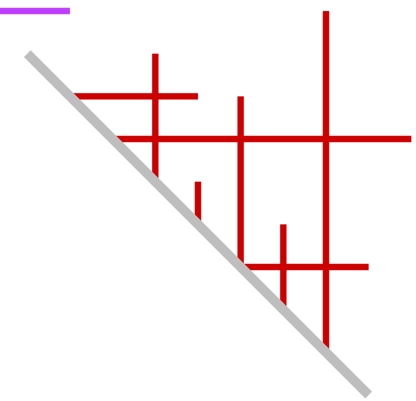

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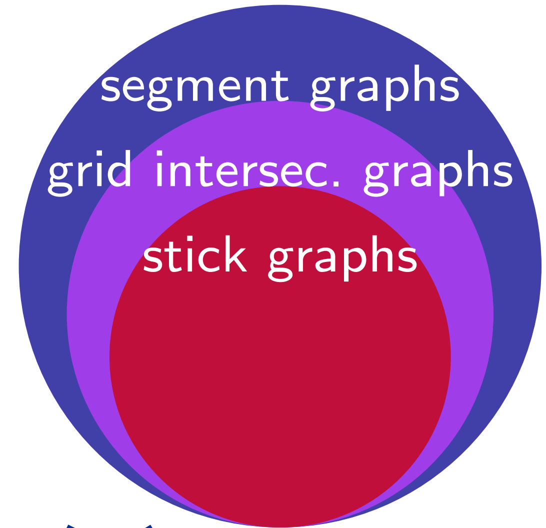

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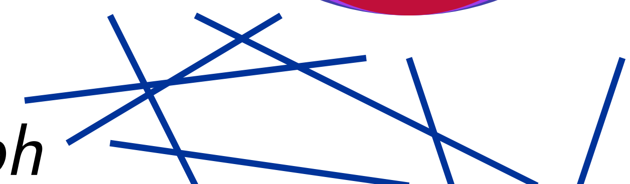

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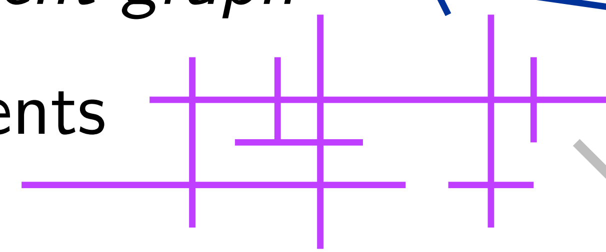

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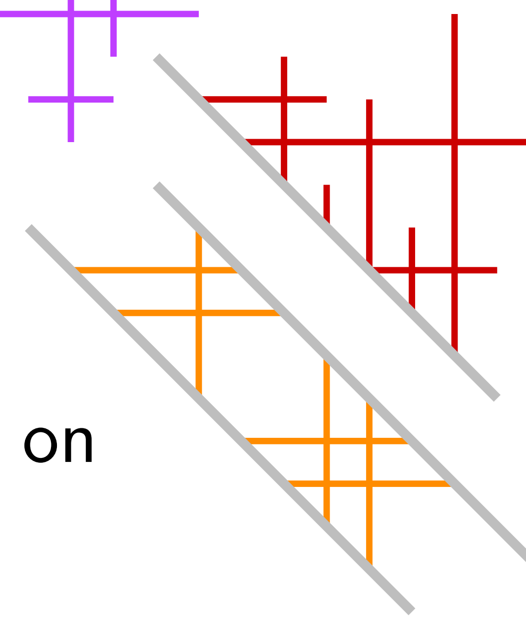

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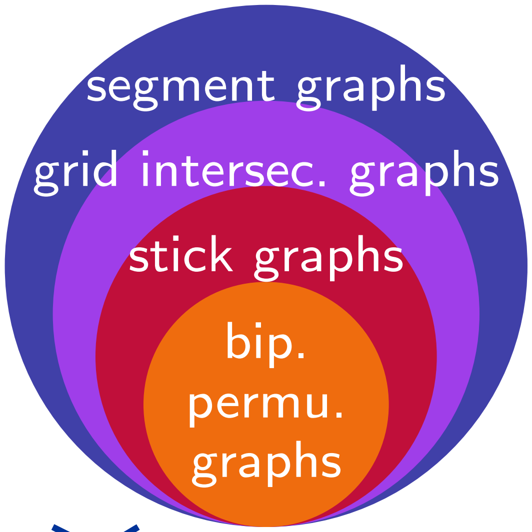

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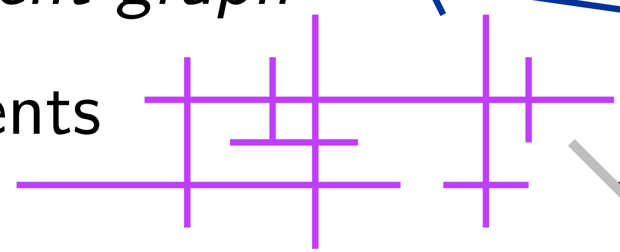

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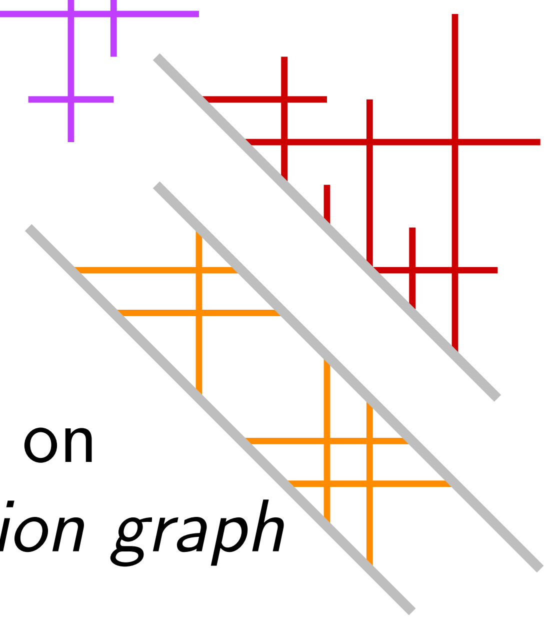

- $\mathcal{S}$: horizontal & vertical segments grounded on a line of slope -1 $\Rightarrow$ G: *stick graph*
- $\mathcal{S}$: horizontal & vertical segments grounded on two parallel lines $\Rightarrow$ G: *bipartite permutation graph*

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Recognition problem:
Decide whether a given graph is an intersection graph.

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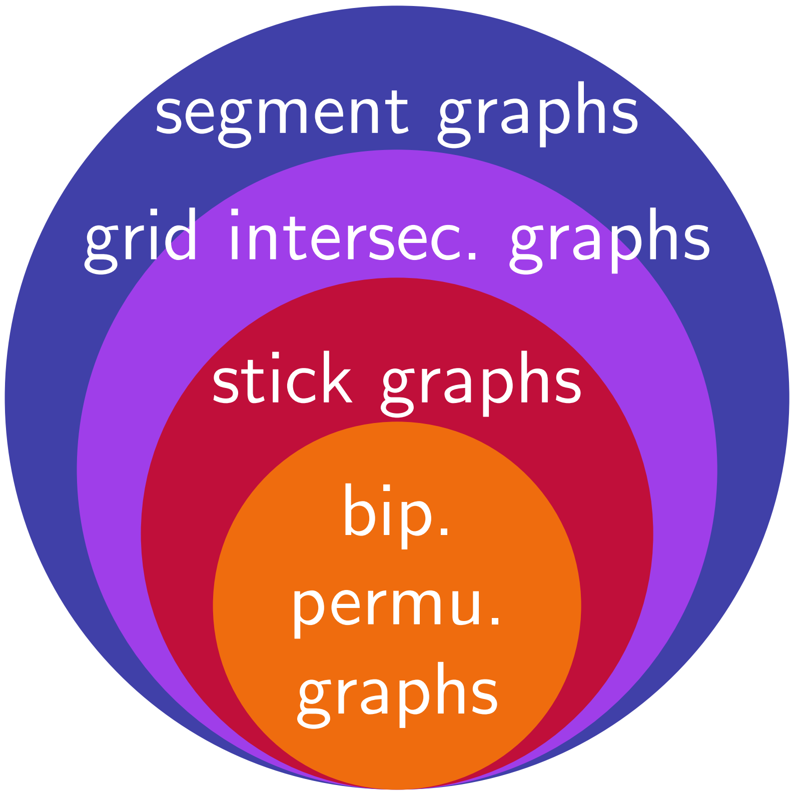
How hard for each class?



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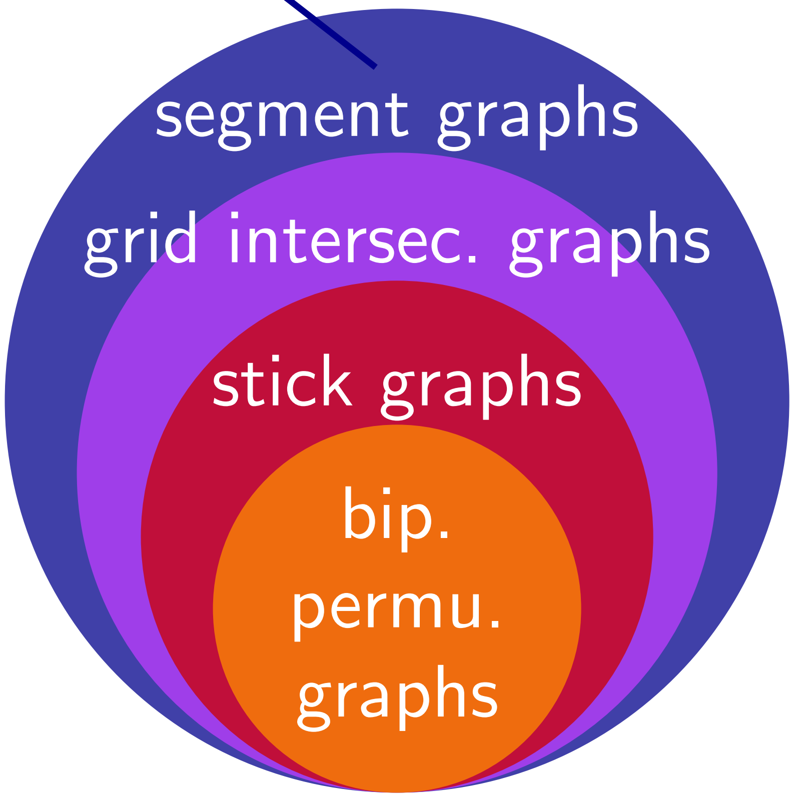

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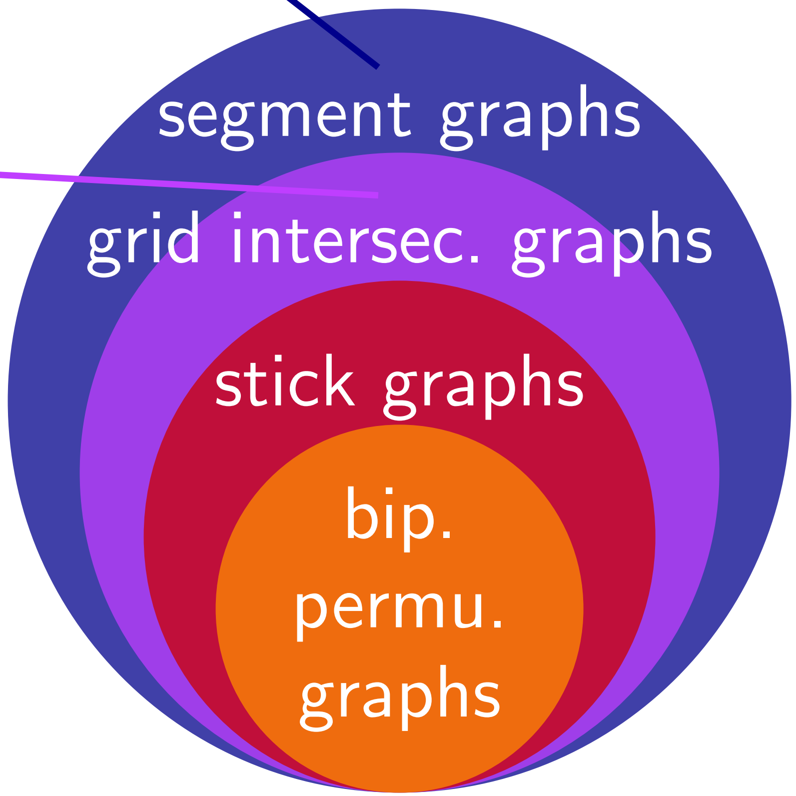

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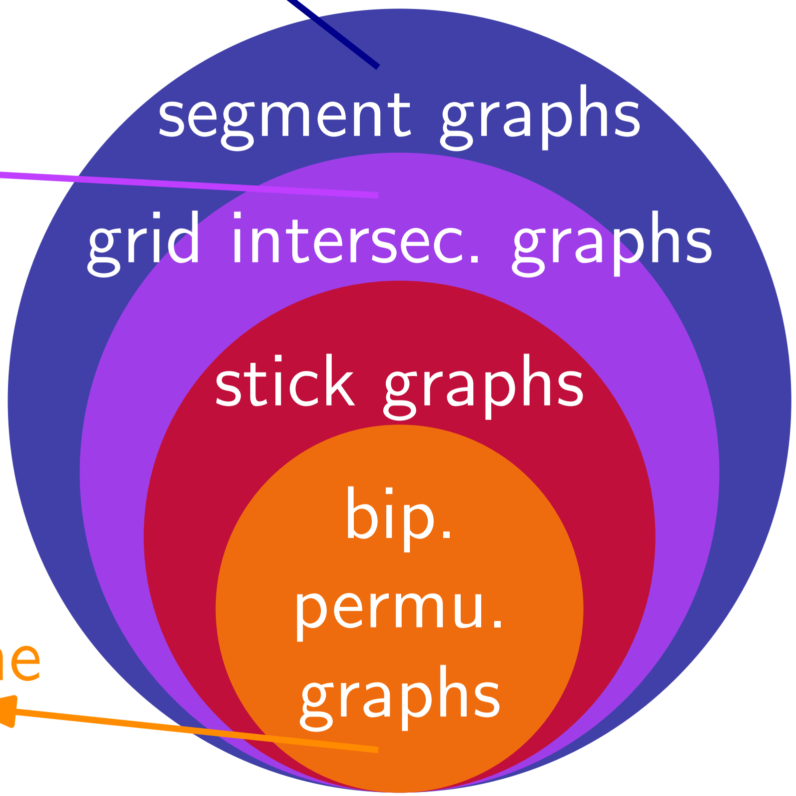

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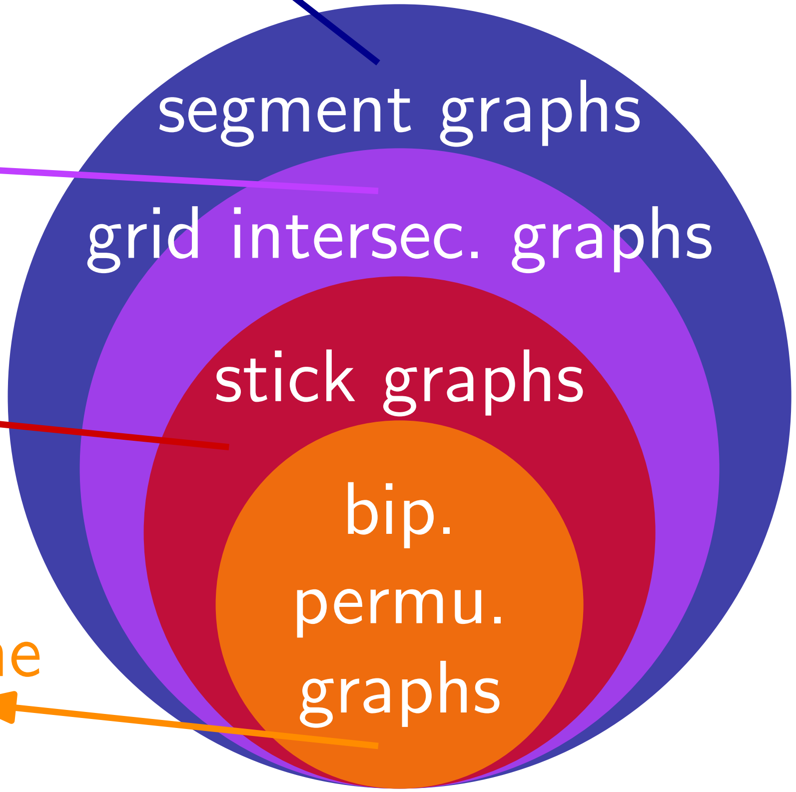

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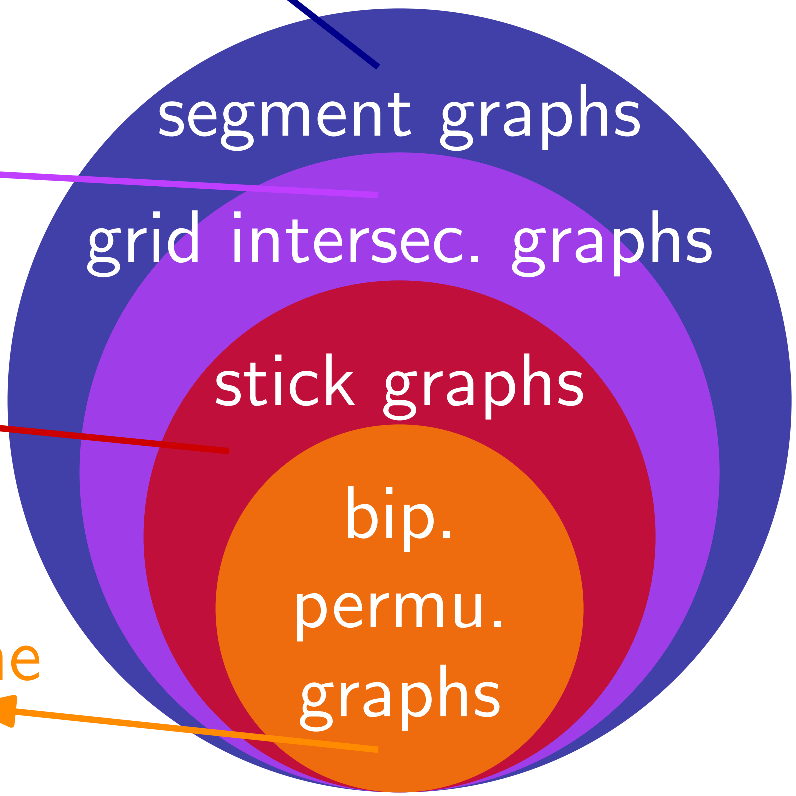

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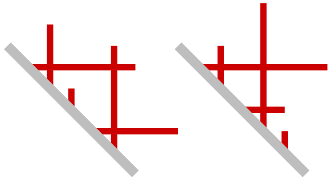

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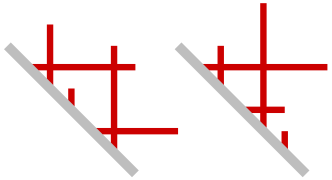

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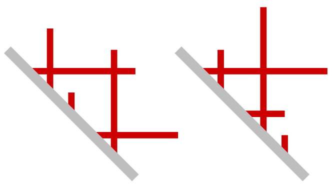

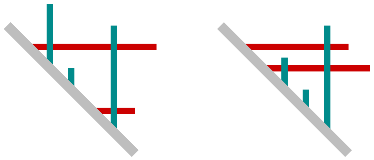

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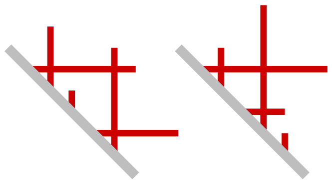

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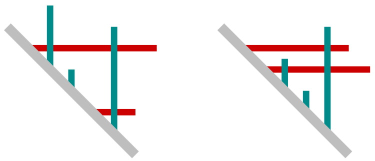

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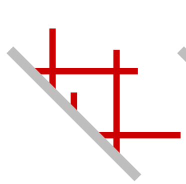

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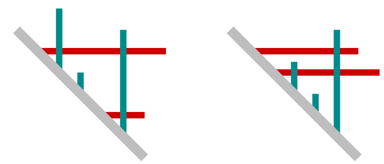

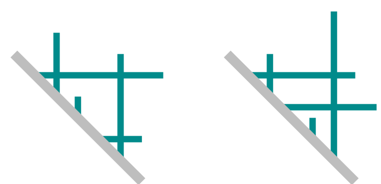

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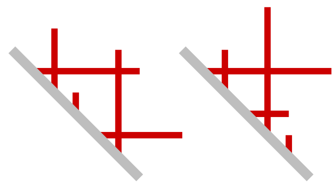

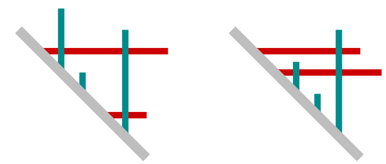

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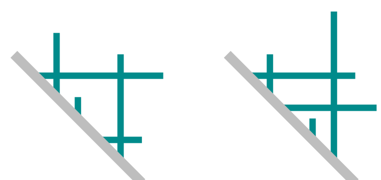

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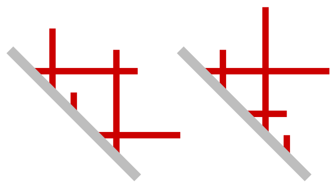

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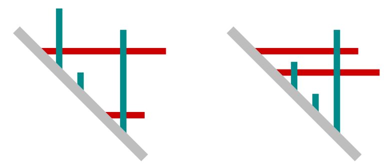

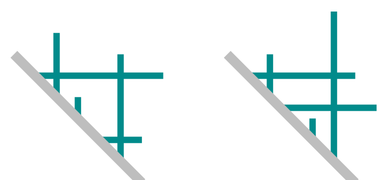

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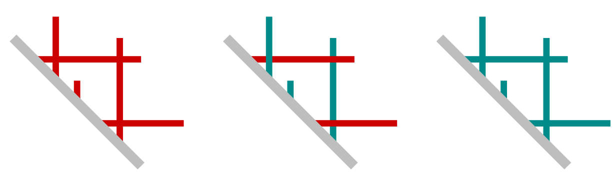

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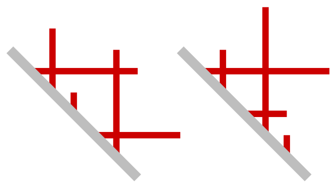

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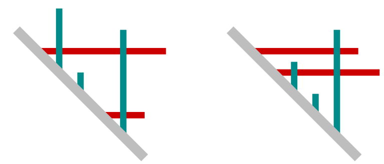

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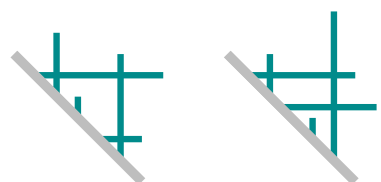

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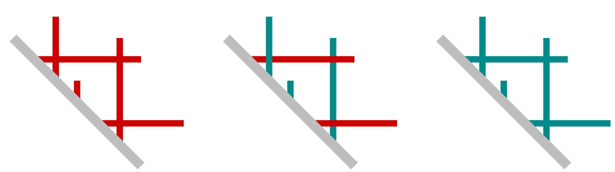

First investigated by De Luca et al. [GD'18]

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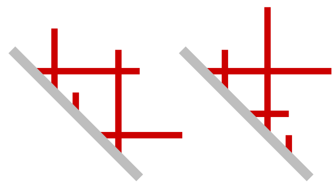

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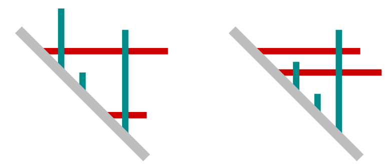

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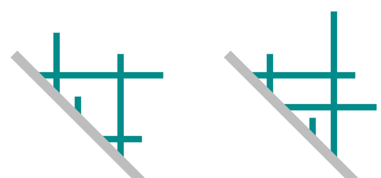

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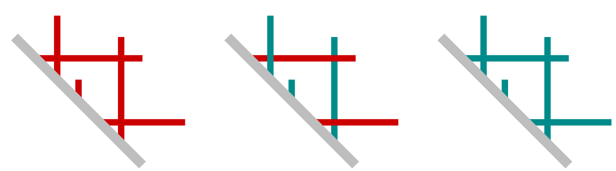

First investigated by De Luca et al. [GD'18]

new

Complexity of Recognition

for a bipartite graph $G = (A \cup B, E)$

$\star$	$\textbf{STICK}_\star$	$\textbf{STICK}_\star^{\textbf{fix}}$
A		
AB		

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$\star$	**STICK**$_\star$	**STICK**$_\star^{\mathbf{fix}}$
	?	**?**
A	**?**[1]	**?**
AB	$O(\lvert A\rvert\lvert B\rvert)$ [De Luca et al. GD'18]	**?**

[1] an $O(|A|^3|B|^3)$ time algorithm proposed by De Luca et al. turned out to be wrong

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our results

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$\star$	**STICK**$_\star$	**STICK**$_\star^{\text{fix}}$
	?	**?** NP-complete
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AB	$O(\lvert A\rvert\lvert B\rvert)$ [De Luca et al. GD'18] $O(\lvert E\rvert)$	**?** in general: NP-complete w/o isolated vtc.: $O((\lvert A\rvert + \lvert B\rvert)^2)$

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our results

$\star$	$\text{STICK}_\star$	$\text{STICK}_\star^{\text{fix}}$
	next ?	? NP-complete
A	?[1] $O(\|A\|\|B\|)$	? NP-complete
AB	$O(\|A\|\|B\|)$ [De Luca et al. GD'18] $O(\|E\|)$	? in general: NP-complete w/o isolated vtc.: $O((\|A\|+\|B\|)^2)$

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next

afterwards

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Algorithm for STICK_{A}

$\star$	**STICK**$_\star$	**STICK**$_\star^{\text{fix}}$
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- Sweep-line along the ordered vertical sticks in A: *enter event* (i) and *exit event* ($i\!\rightarrow$) for each $a_i \in A$

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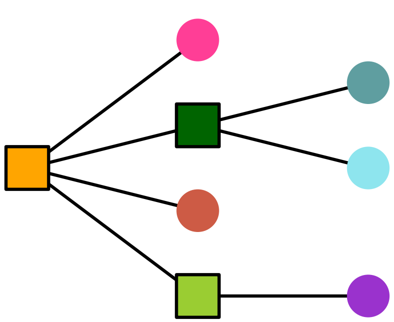

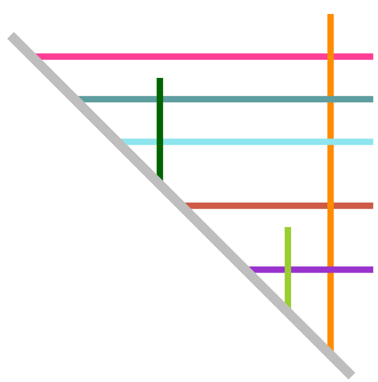

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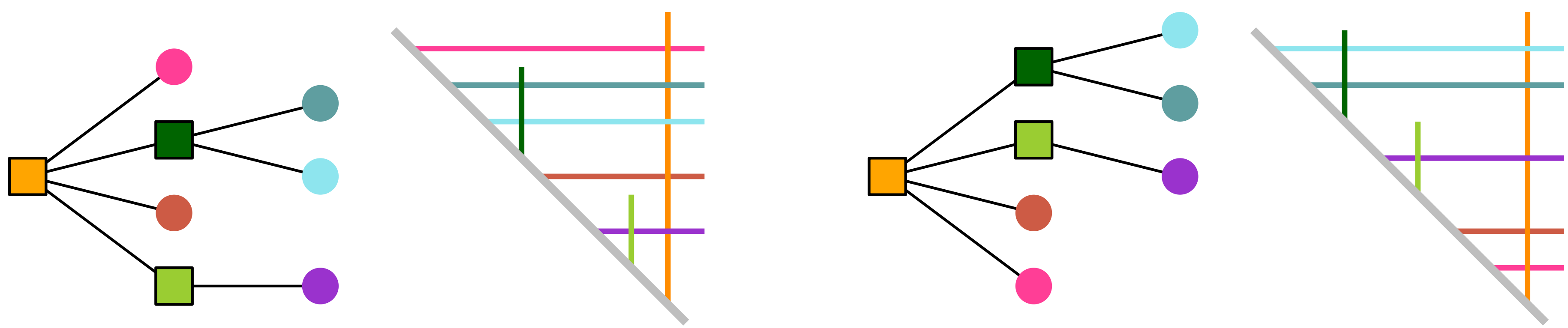
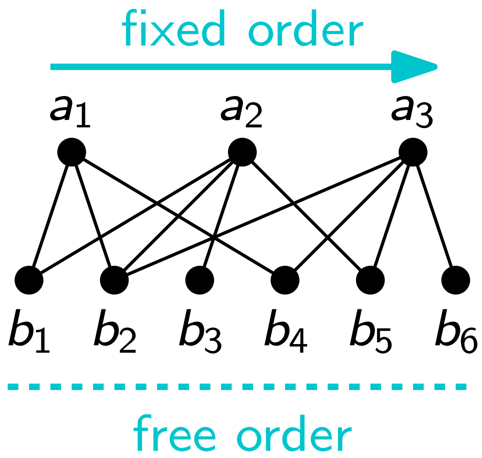


Example for STICK_A

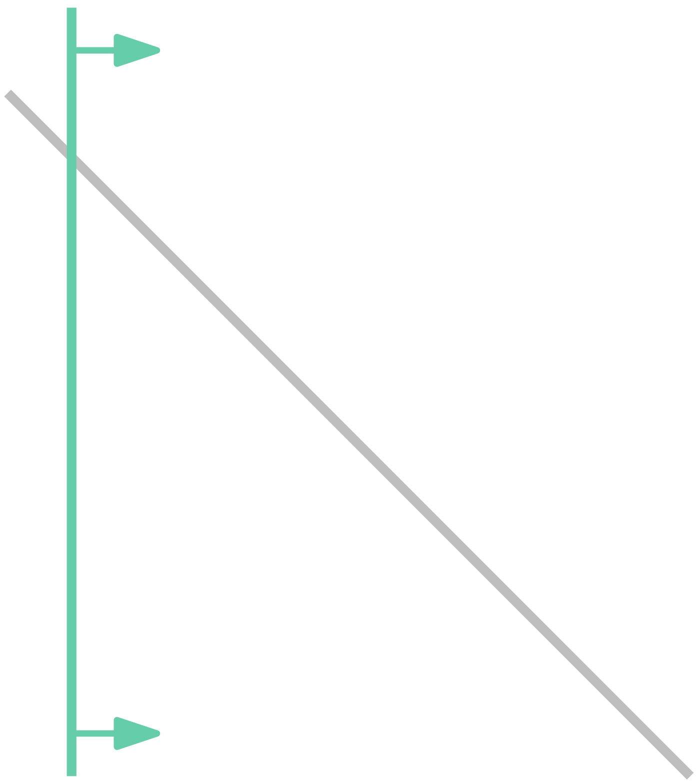
$i = 0$ Event: *Start* $B^0 = \emptyset$

G:



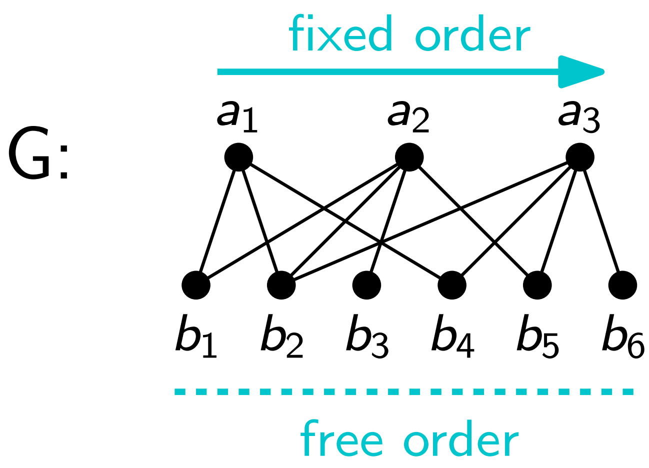
G^0 :

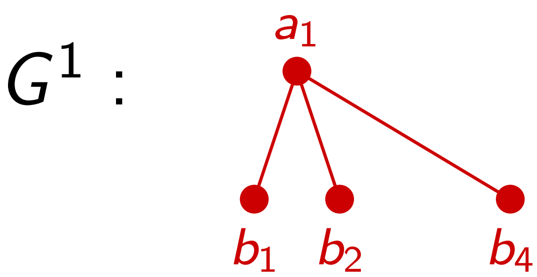
$\mathcal{T}^0$:

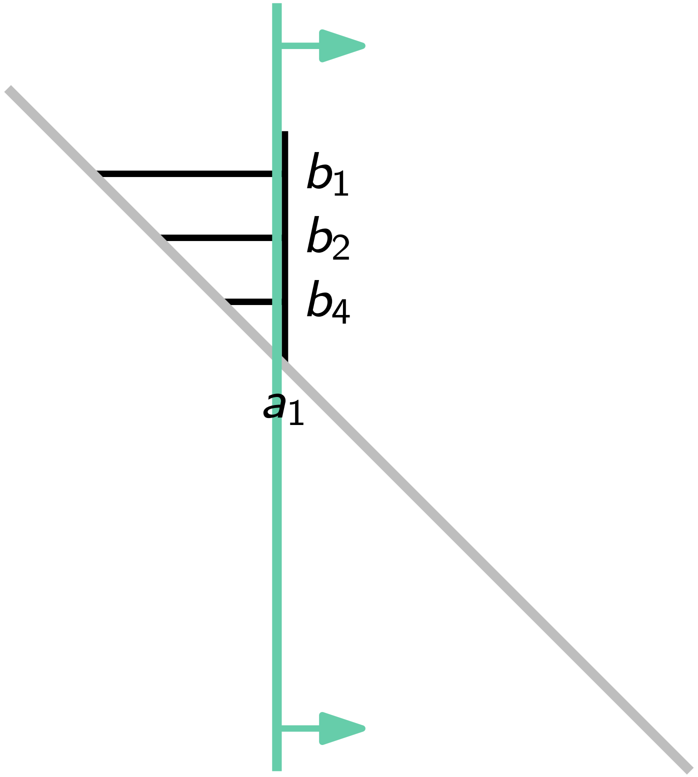


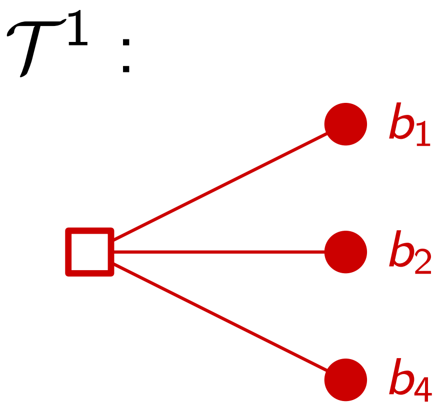
Example for STICK$_A$

$i = 1$ Event: 1 $B^1 = \{b_1, b_2, b_4\}$



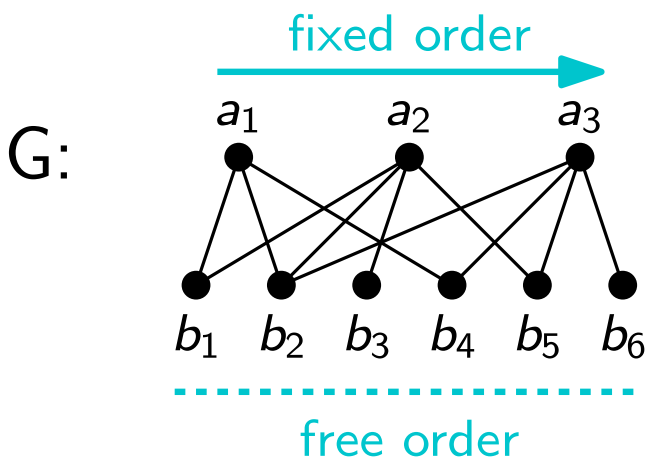


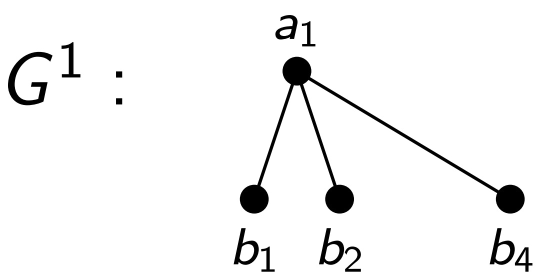


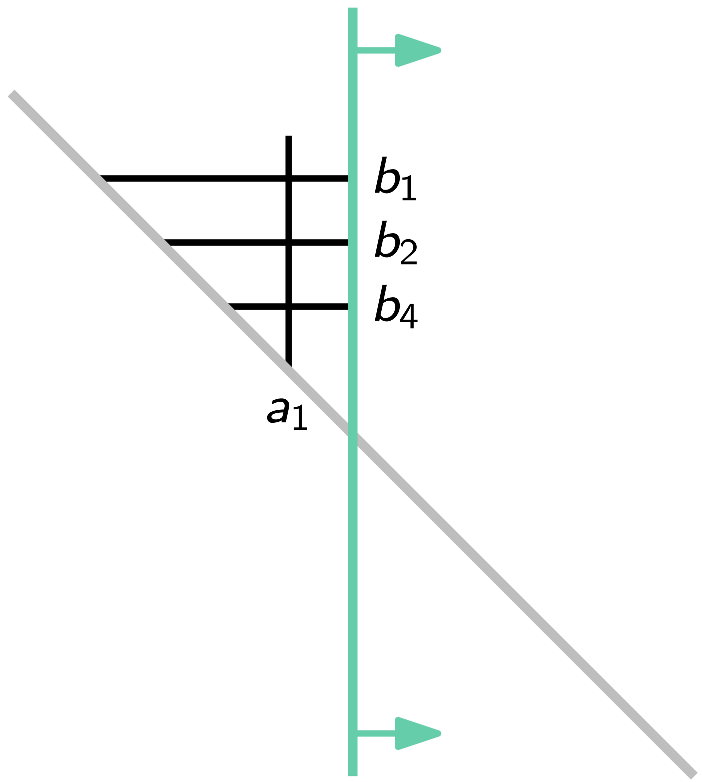


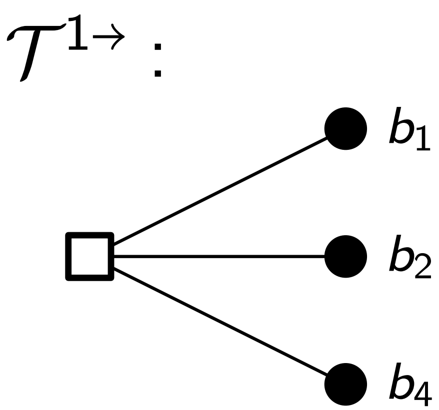
Example for STICK$_A$

$i = 1$ Event: $1{\rightarrow}$ $B^{1\rightarrow} = \{b_1, b_2, b_4\}$









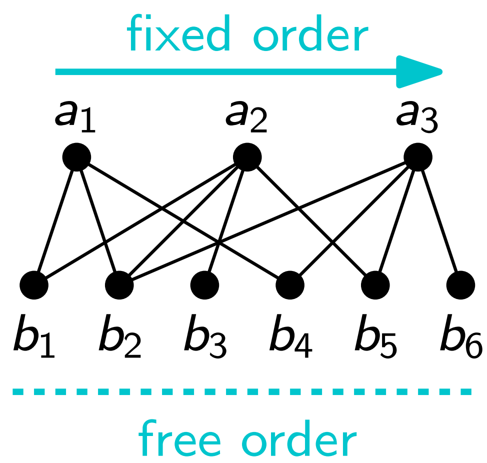
Example for STICK$_A$

$i = 2$

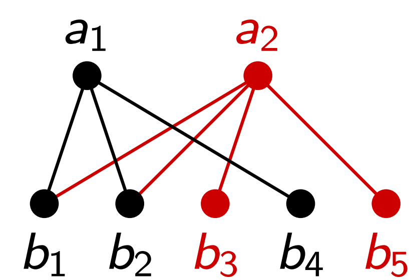
Event: 2

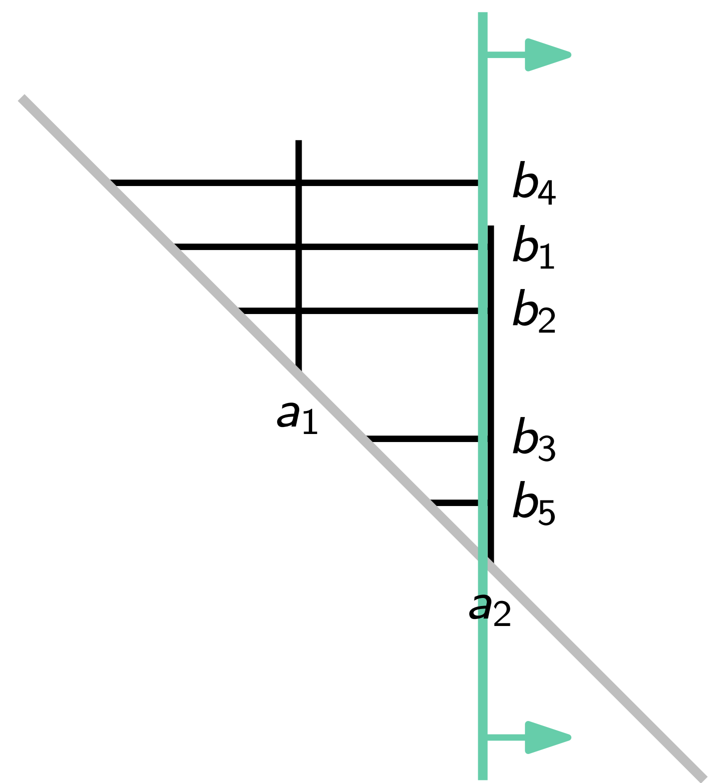
$B^2 = \{b_1, b_2, b_3, b_4, b_5\}$

G:

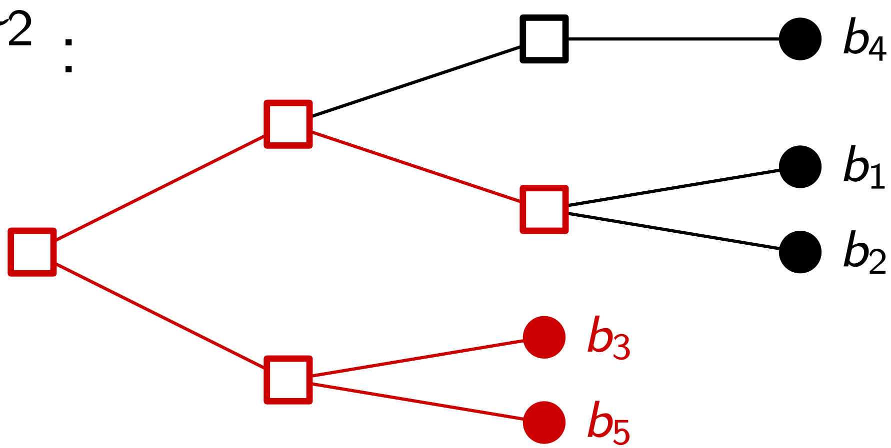


G^2 :





$\mathcal{T}^2$:



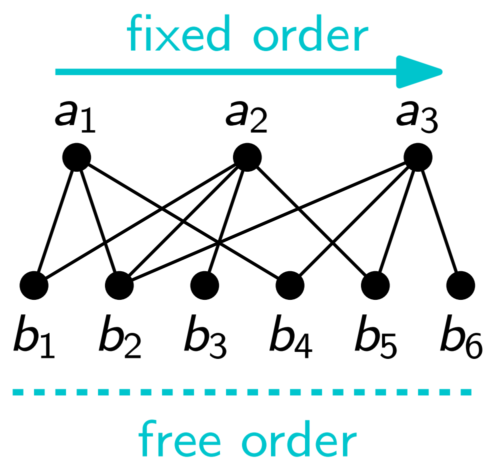
Example for STICK_A

$i = 2$

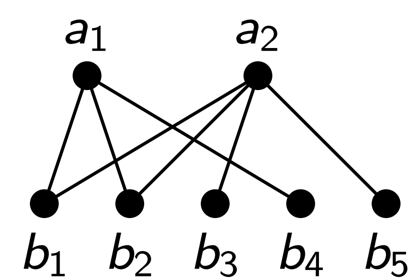
Event: $2{\rightarrow}$

$B^{2\rightarrow} = \{b_1, b_2, b_3, b_4, b_5\}$

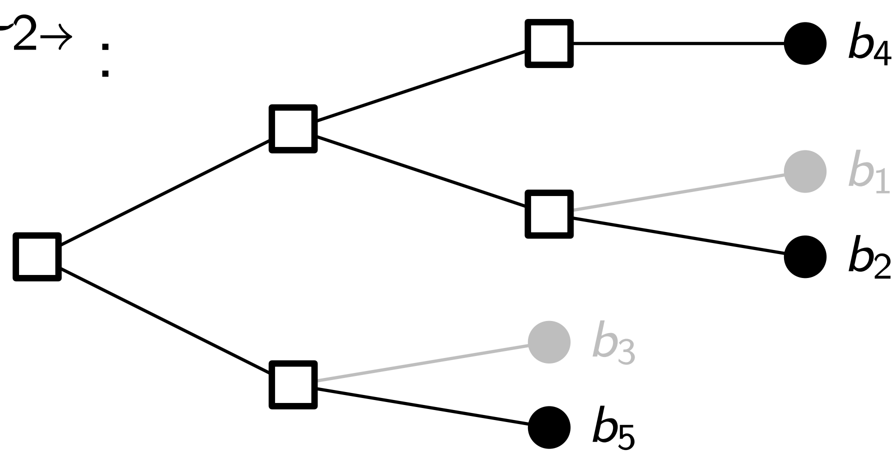
G:

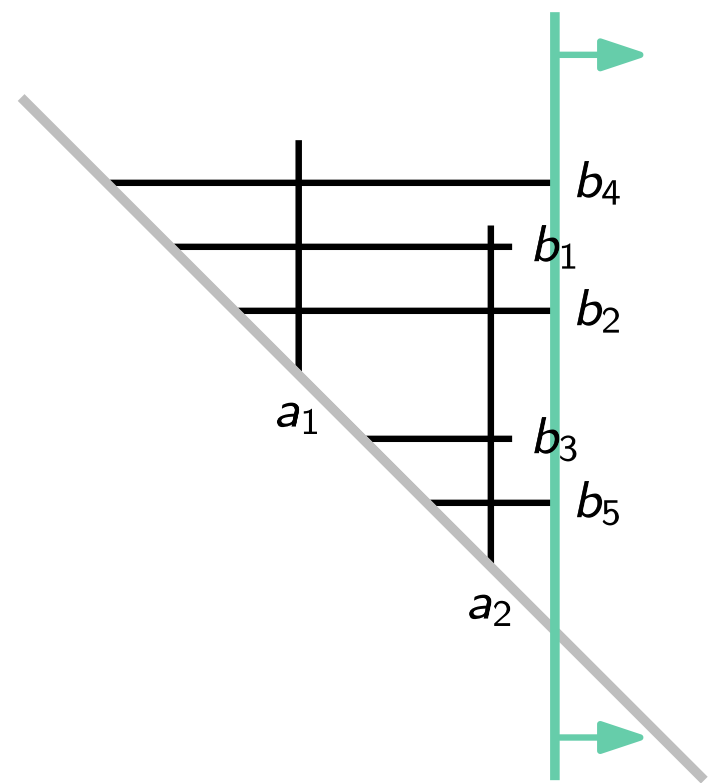


G^2 :



$\mathcal{T}^{2\rightarrow}$:

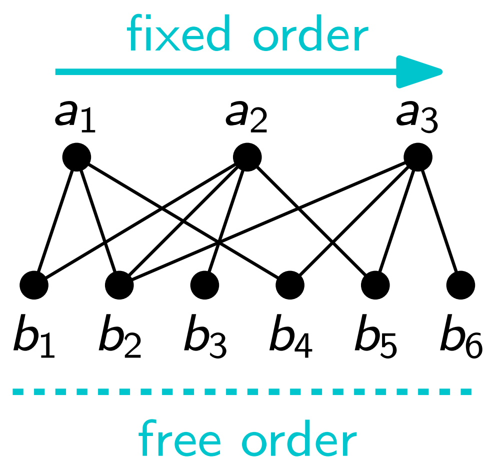




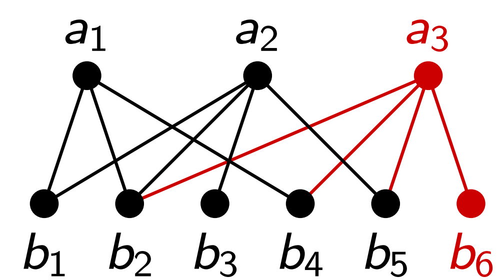
Example for STICK_A

$i = 3$ Event: 3 $B^3 = \{b_2, b_4, b_5, b_6\}$

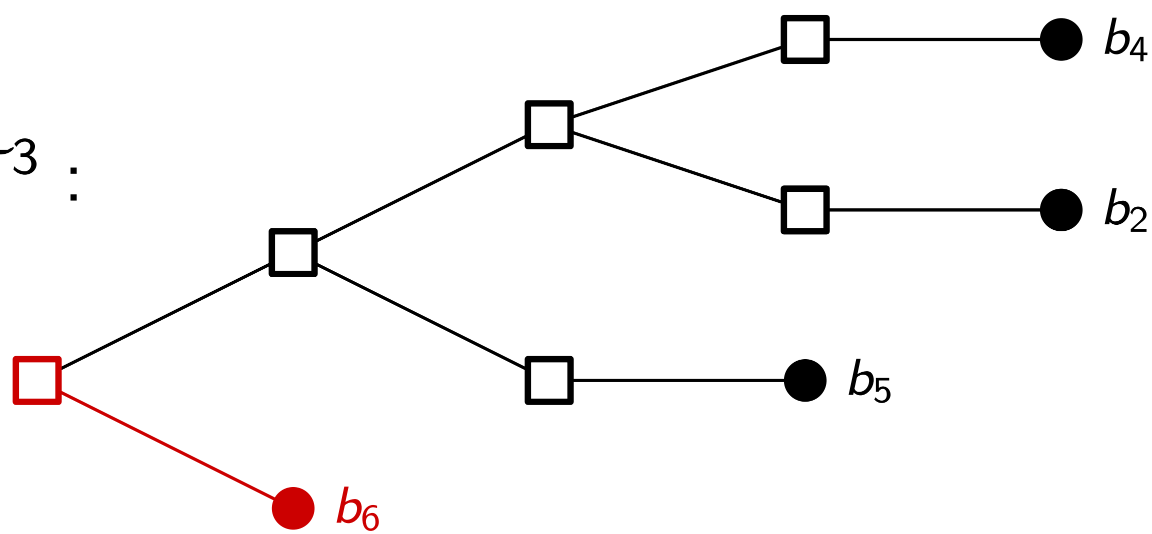
G:

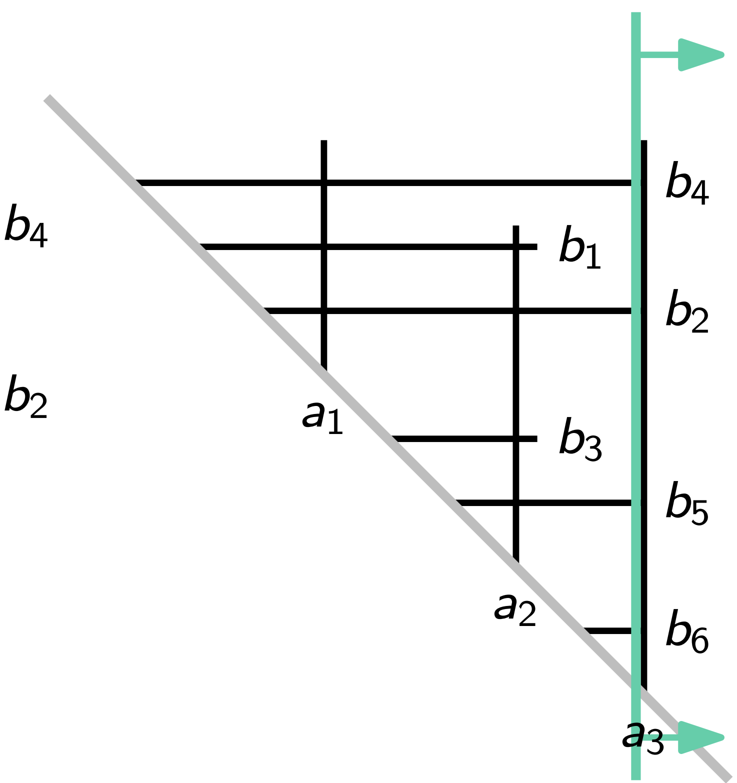


G^3 :



$\mathcal{T}^3$:

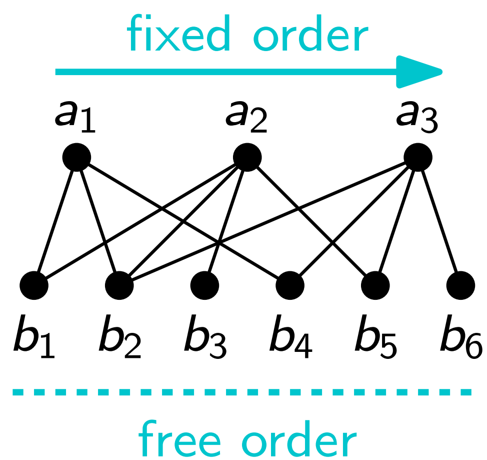




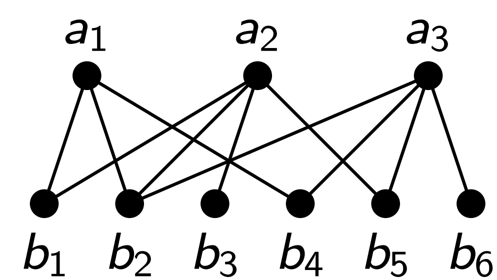
Example for STICK_A

$i = 3$ Event: $3{\rightarrow}$ $B^{3\rightarrow} = \{b_2, b_4, b_5, b_6\}$

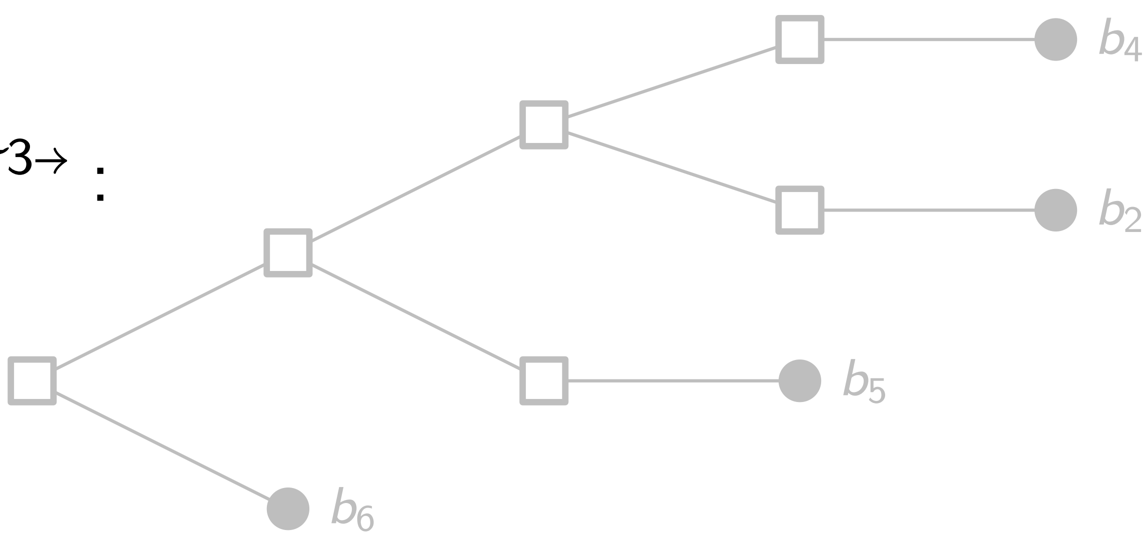
G:

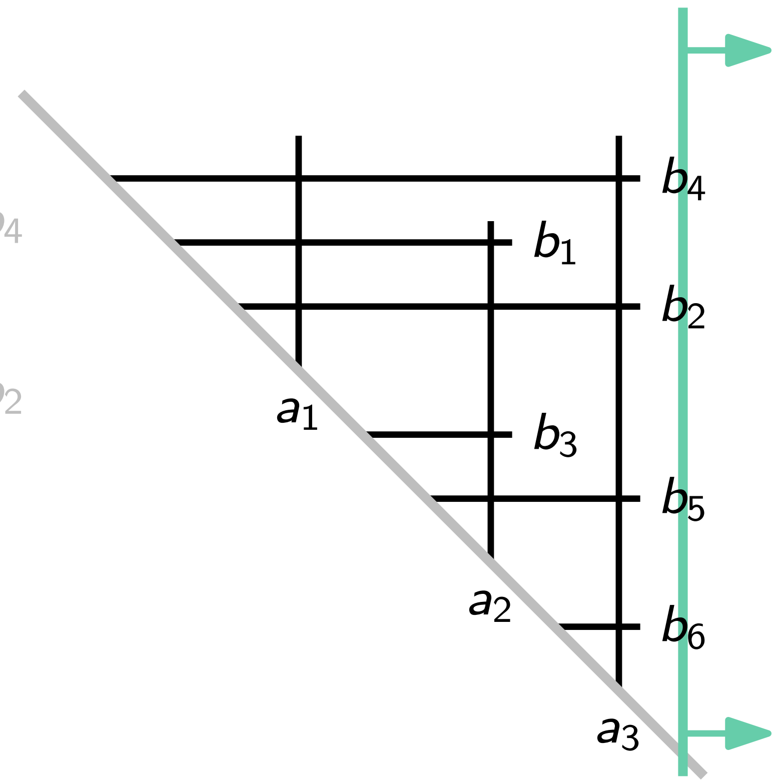


G^3 :



$\mathcal{T}^{3\rightarrow}$:

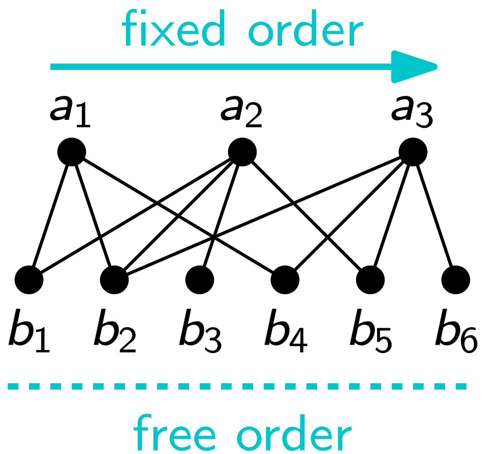


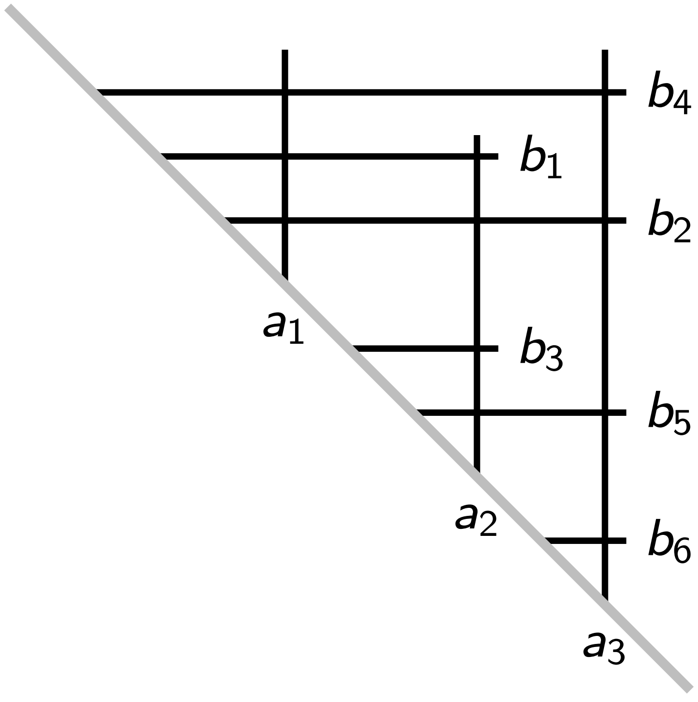


Example for STICK_A

$i = 3$ Event: *End*

G:





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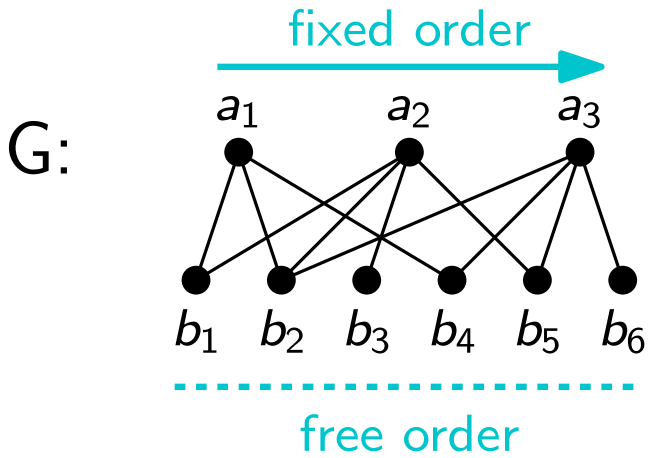
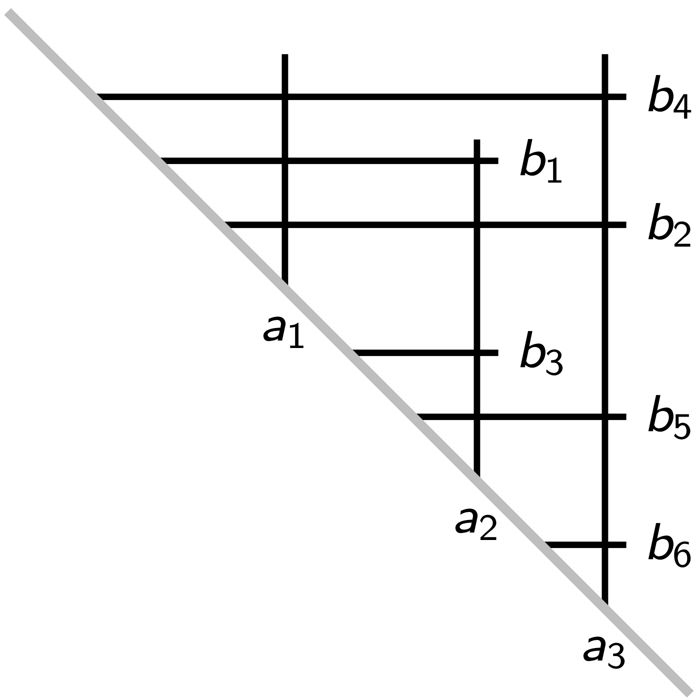


Runtime in $O(|A| \cdot |B|)$



$\mathsf{STICK}_{\mathsf{AB}}^{\mathsf{fix}}$ with isolated vertices

$\star$	$\mathsf{STICK}_\star$	$\mathsf{STICK}_\star^{\mathsf{fix}}$
	?	NP-complete
A	$O(\|A\|\|B\|)$	NP-complete
AB	$O(\|E\|)$	in general: NP-complete w/o isolated vtc.: $O((\|A\|+\|B\|)^2)$

Hardness of STICK$^{\mathsf{fix}}_{\mathsf{AB}}$

- NP-hardness by reduction from MONOTONE-3-SAT

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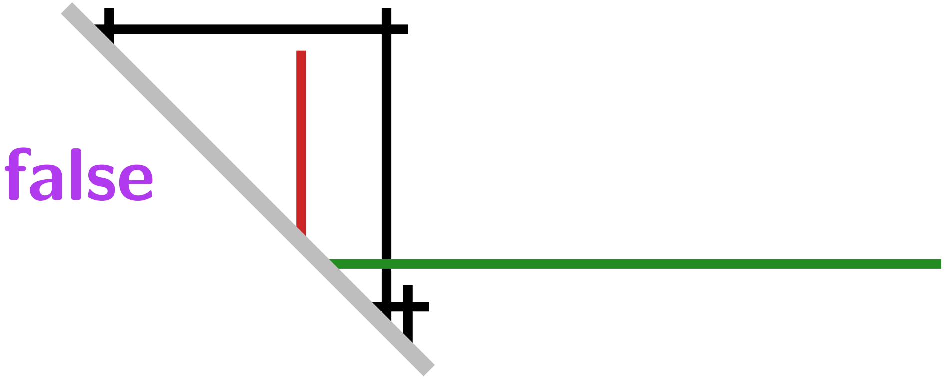
- Variable gadget:



Hardness of STICK$^{\text{fix}}_{\text{AB}}$

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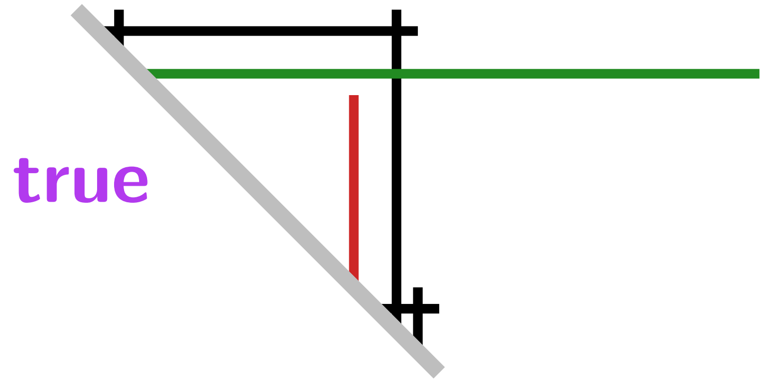

Hardness of STICK$^{\text{fix}}_{\text{AB}}$

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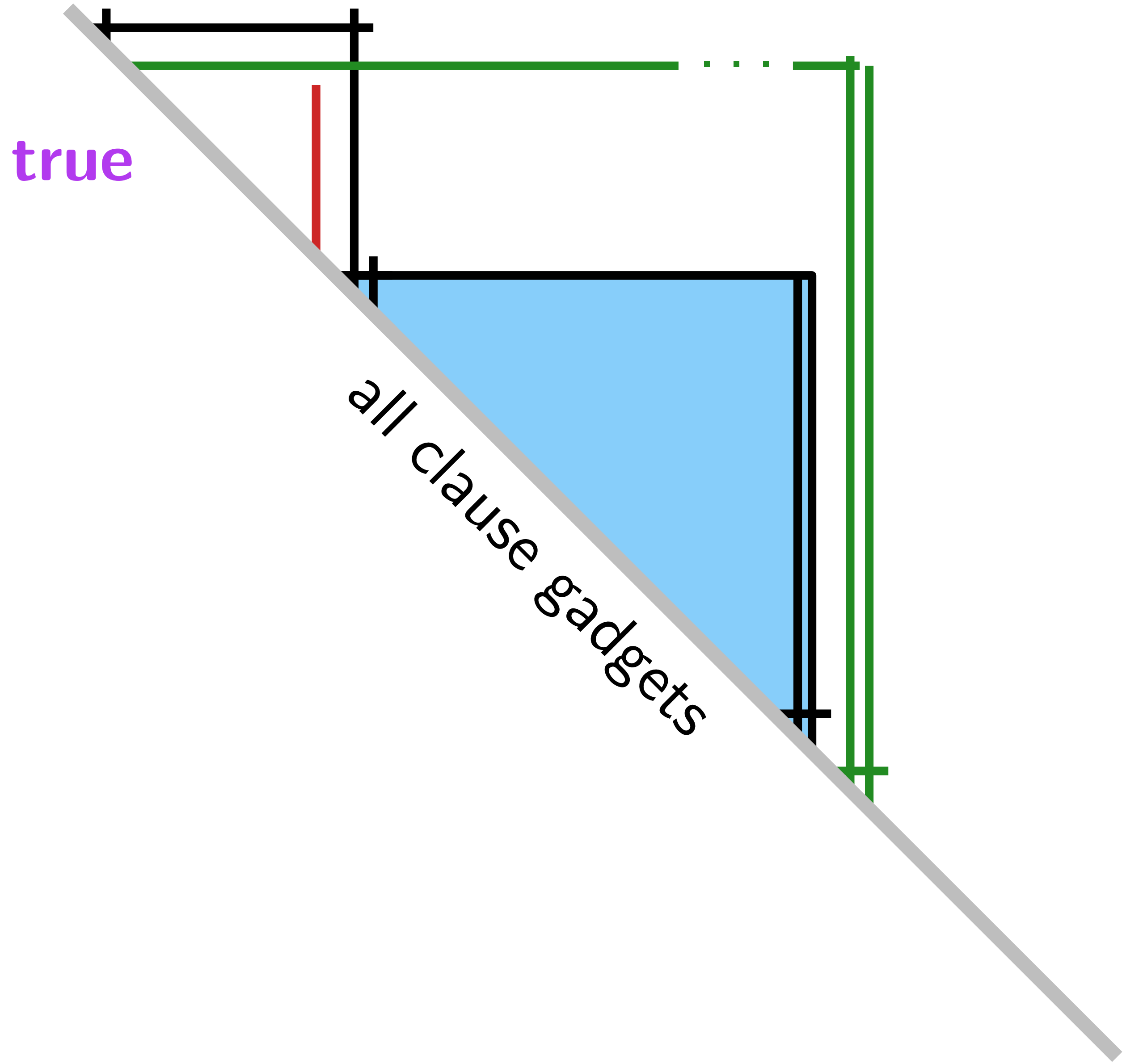

Hardness of $\text{STICK}_{\text{AB}}^{\text{fix}}$

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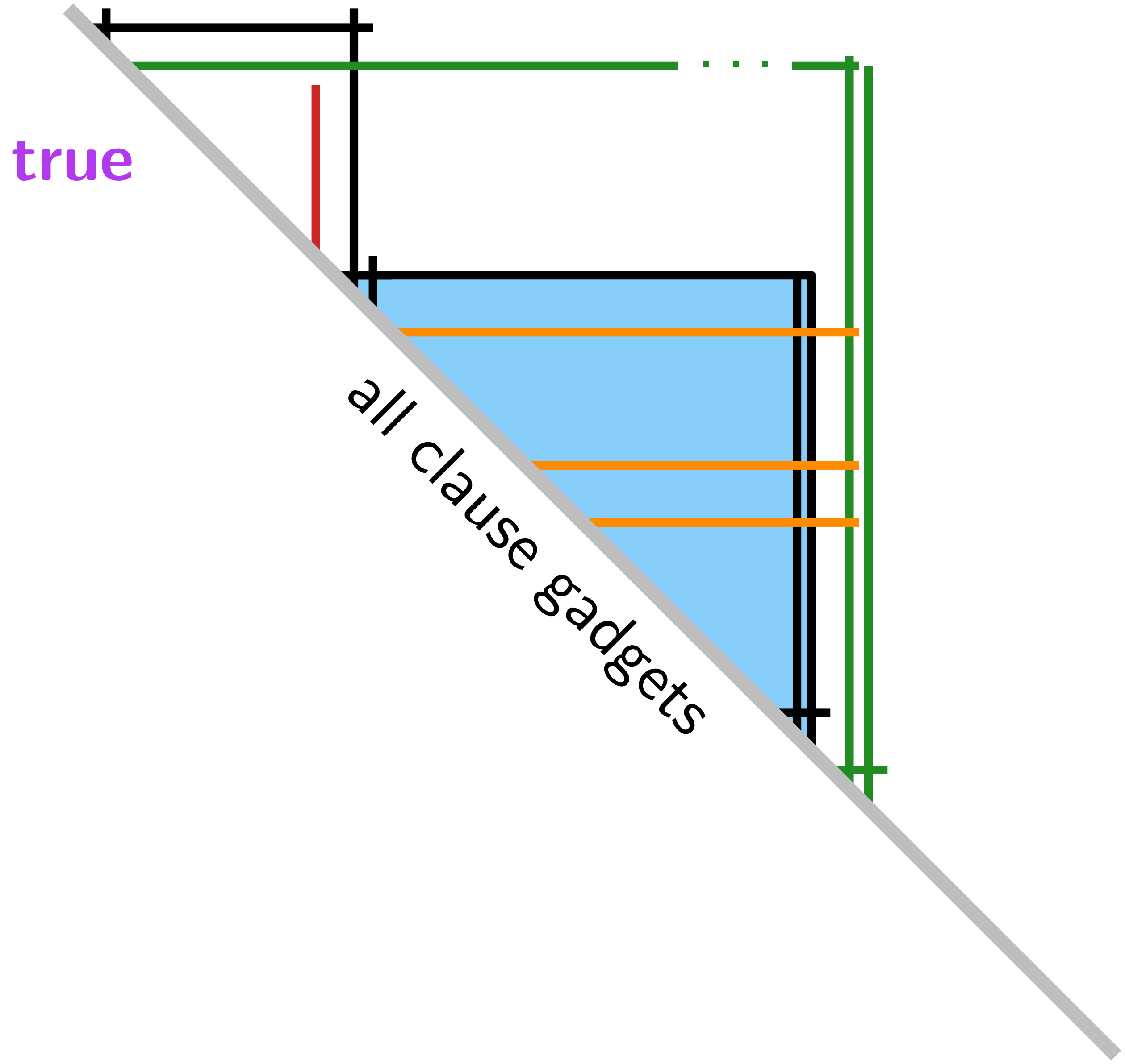
- Variable gadget:



Hardness of STICK$^{\text{fix}}_{\text{AB}}$

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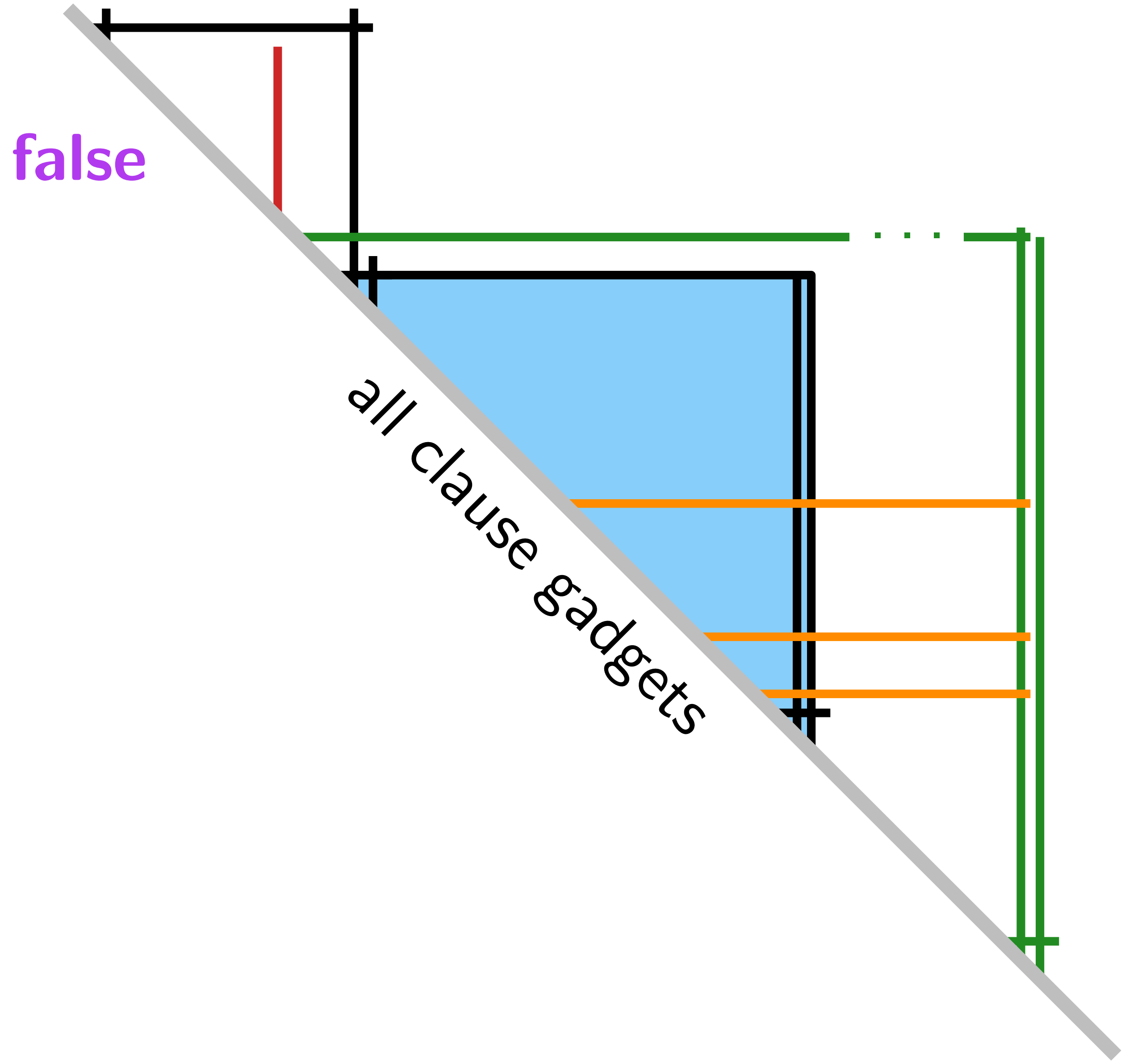
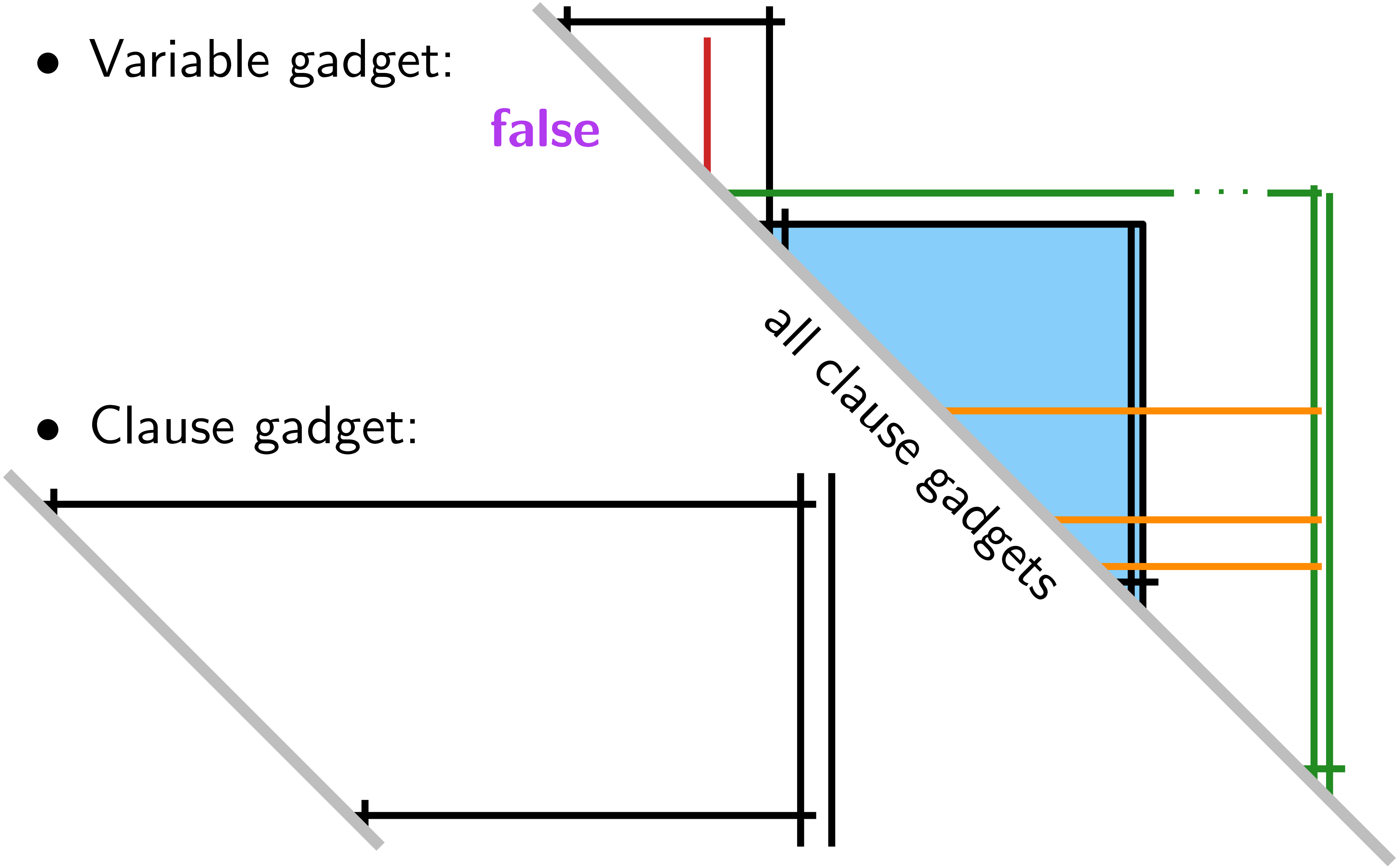
- Variable gadget:



Hardness of $\text{STICK}^{\text{fix}}_{\text{AB}}$

- NP-hardness by reduction from MONOTONE-3-SAT
- Variable gadget:
- Clause gadget:



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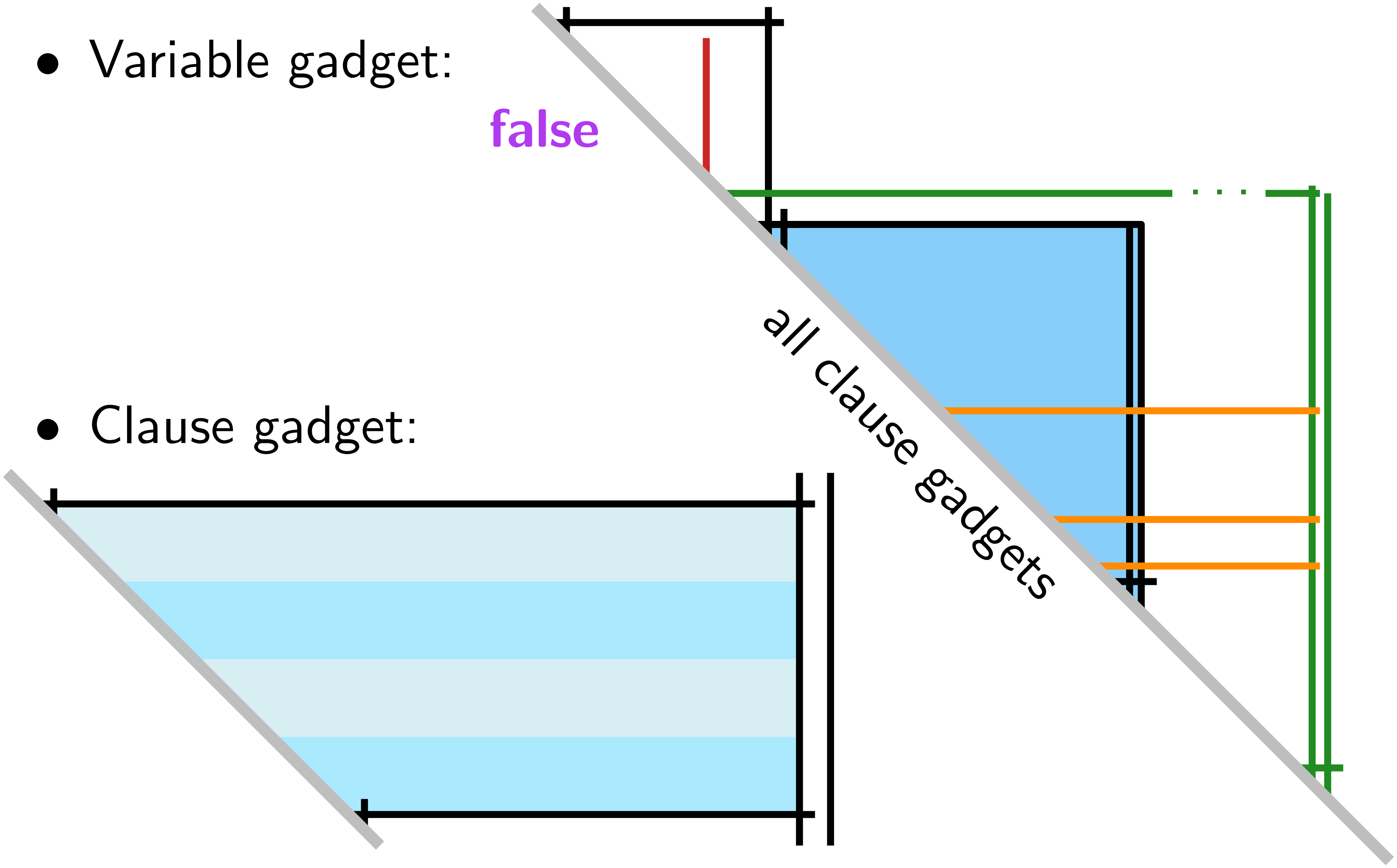

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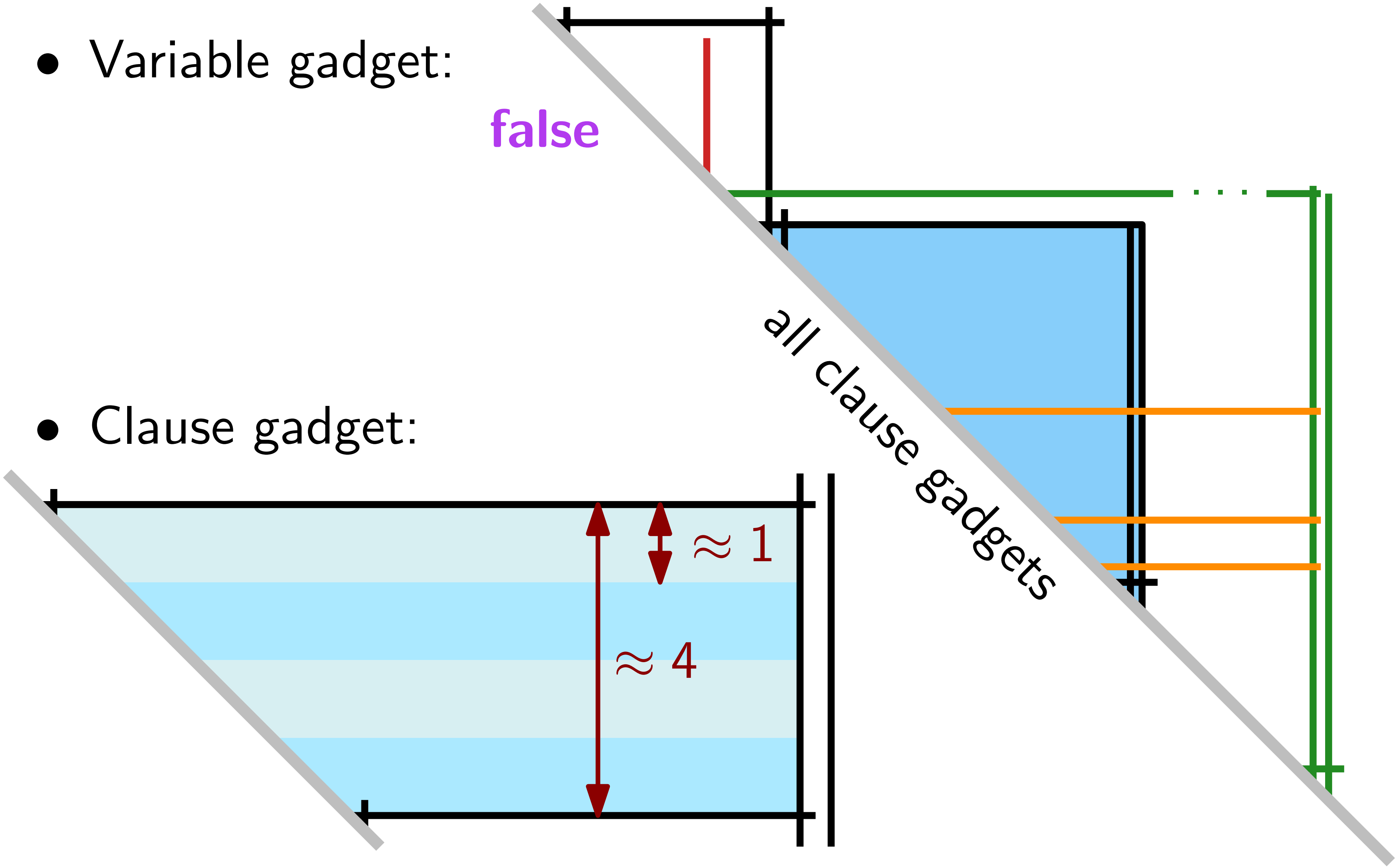

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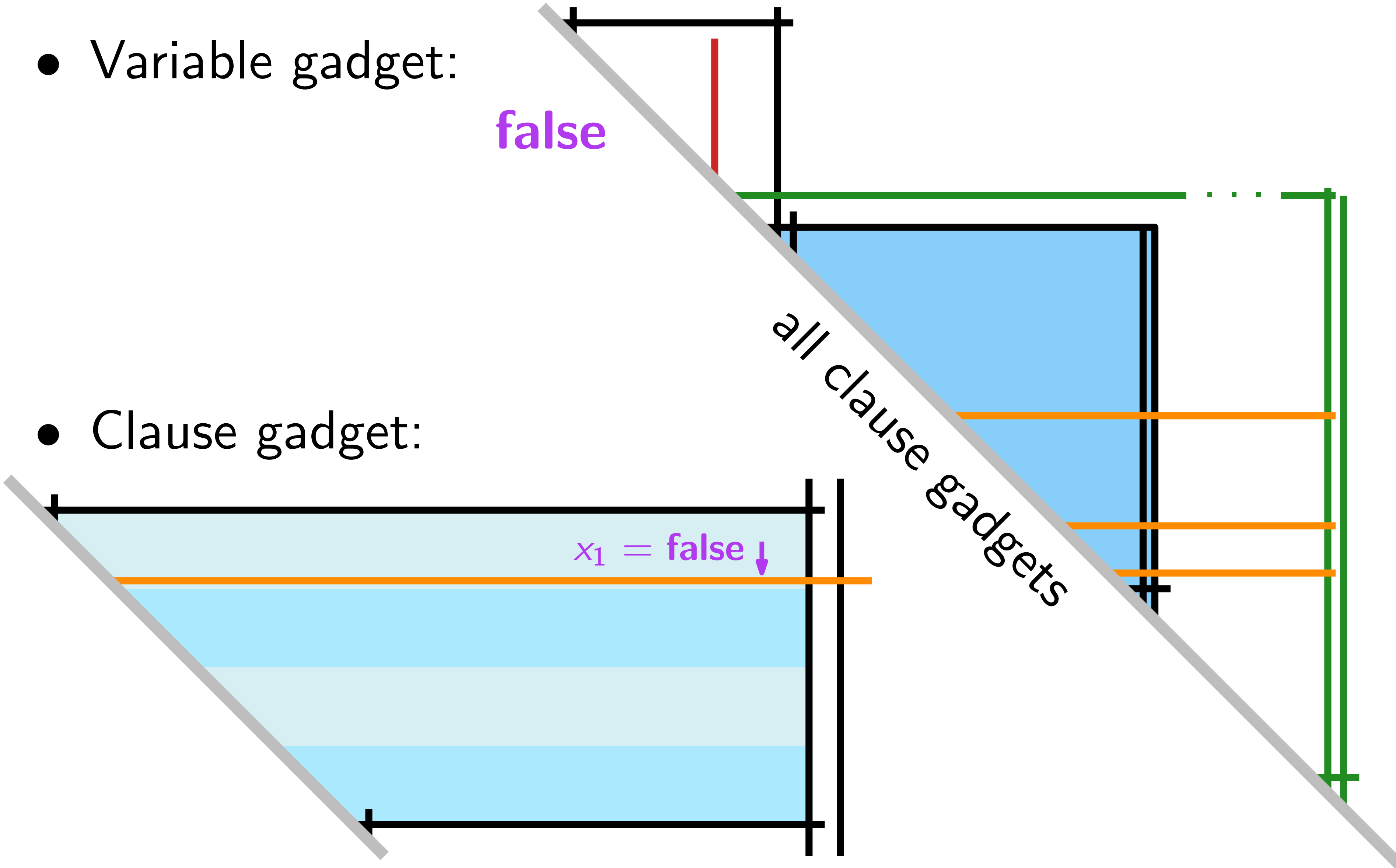

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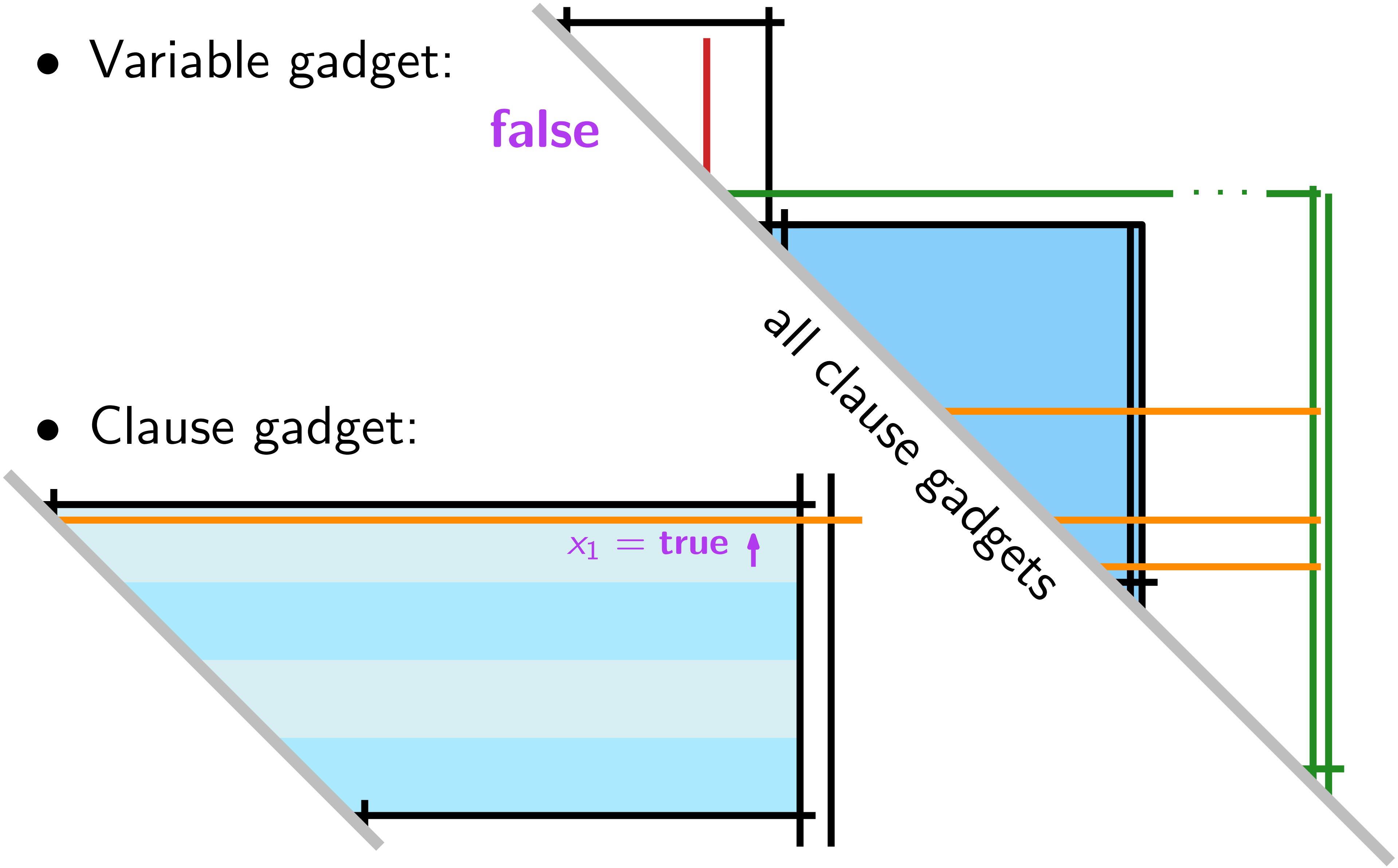

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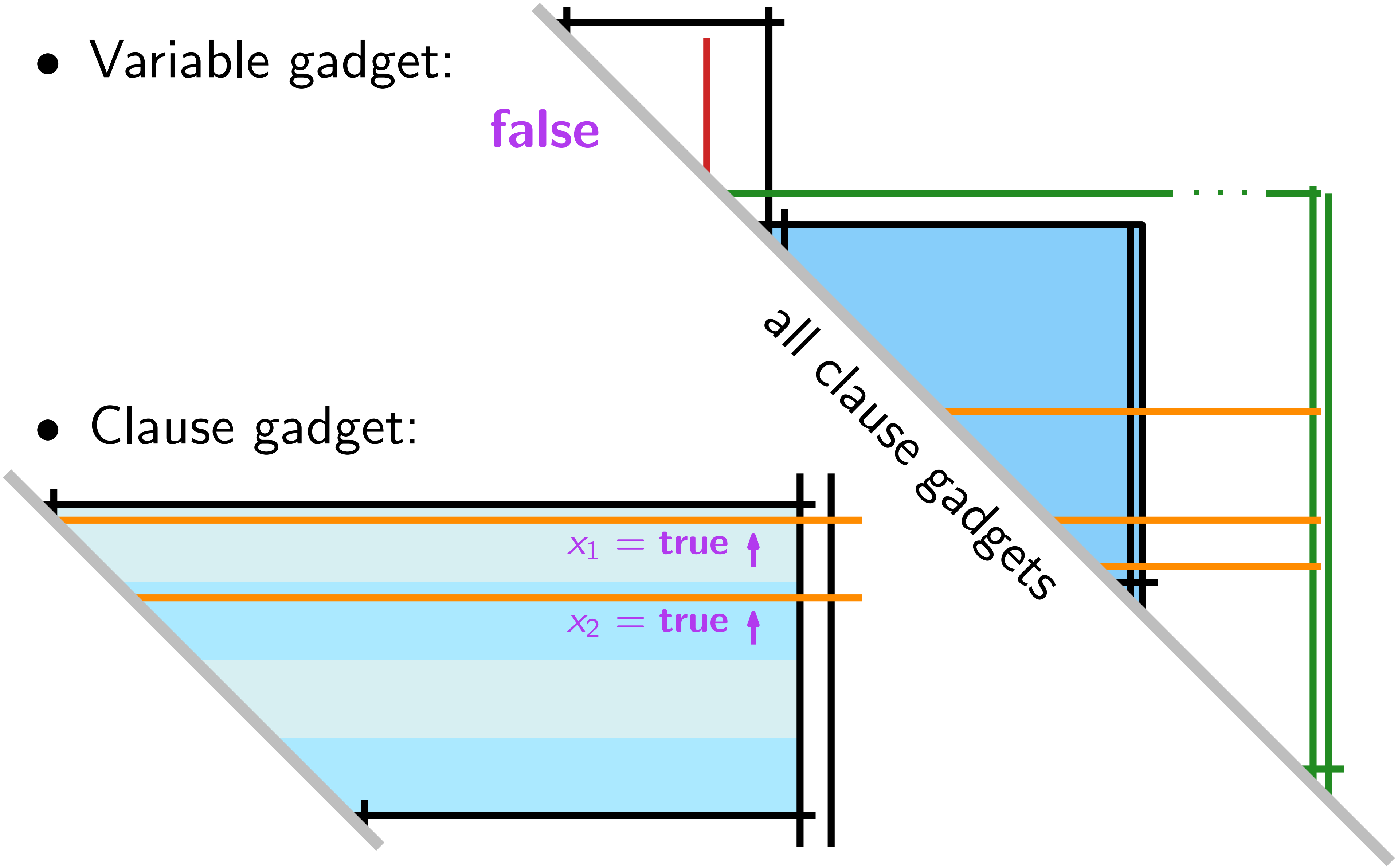

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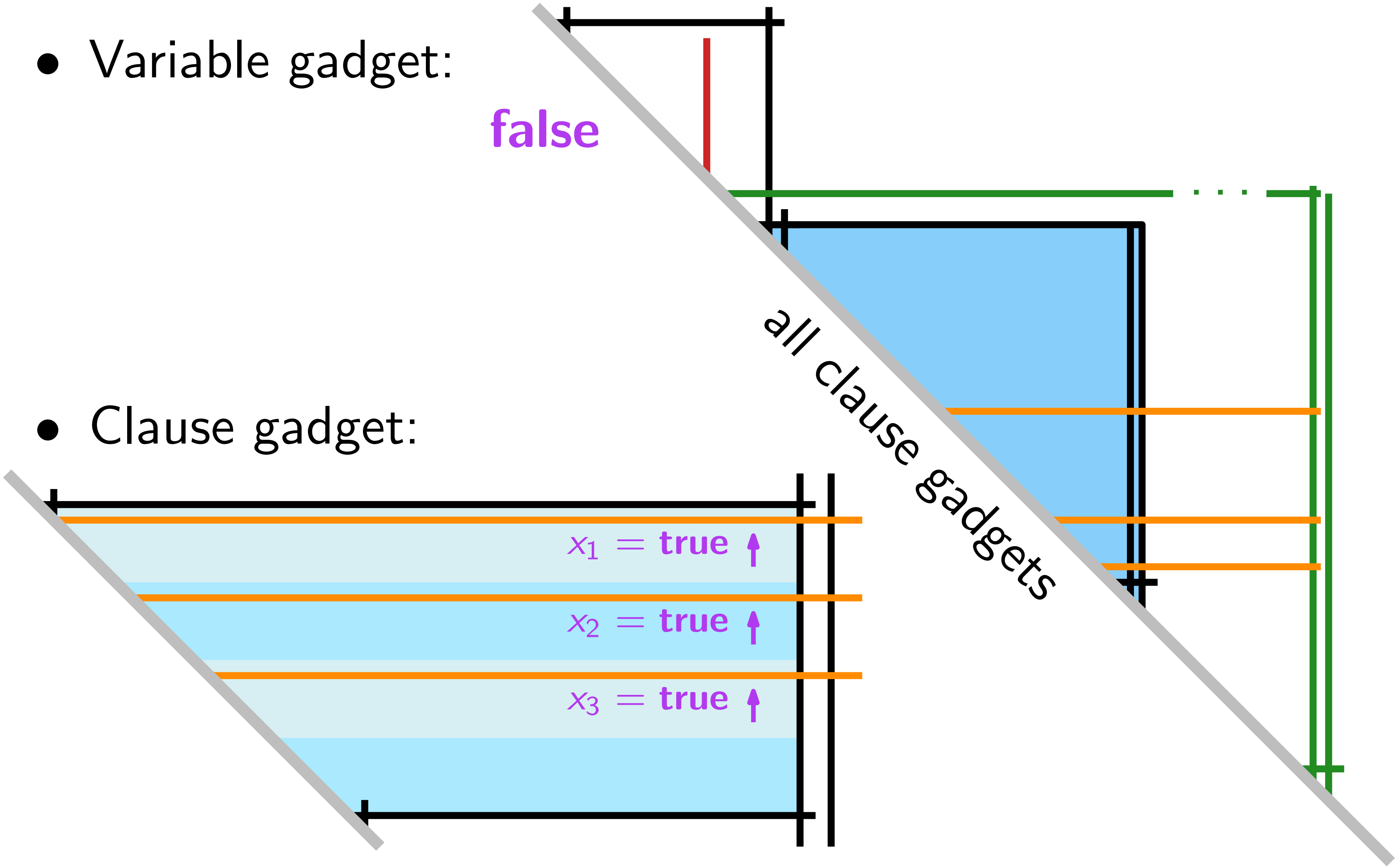

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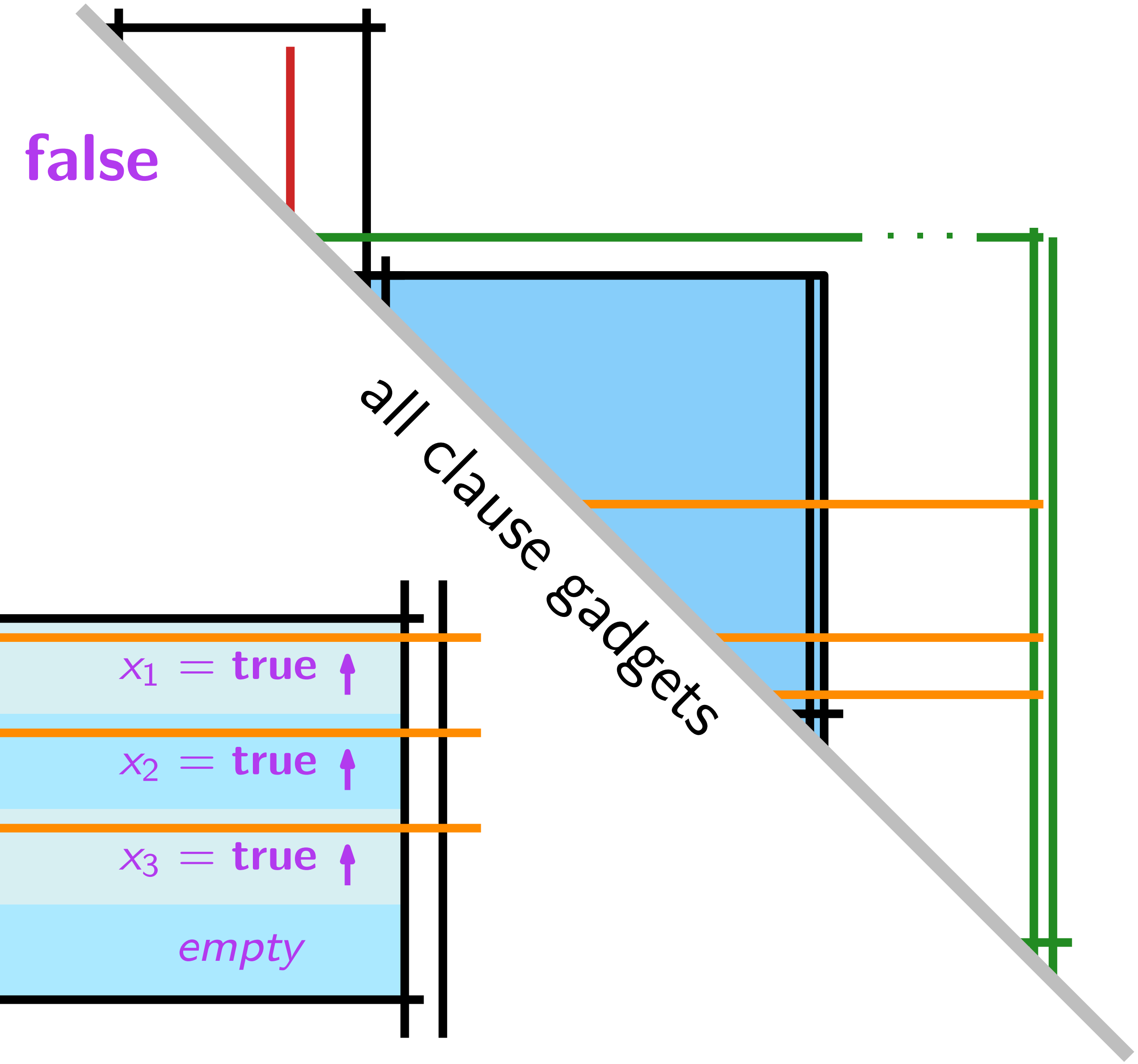

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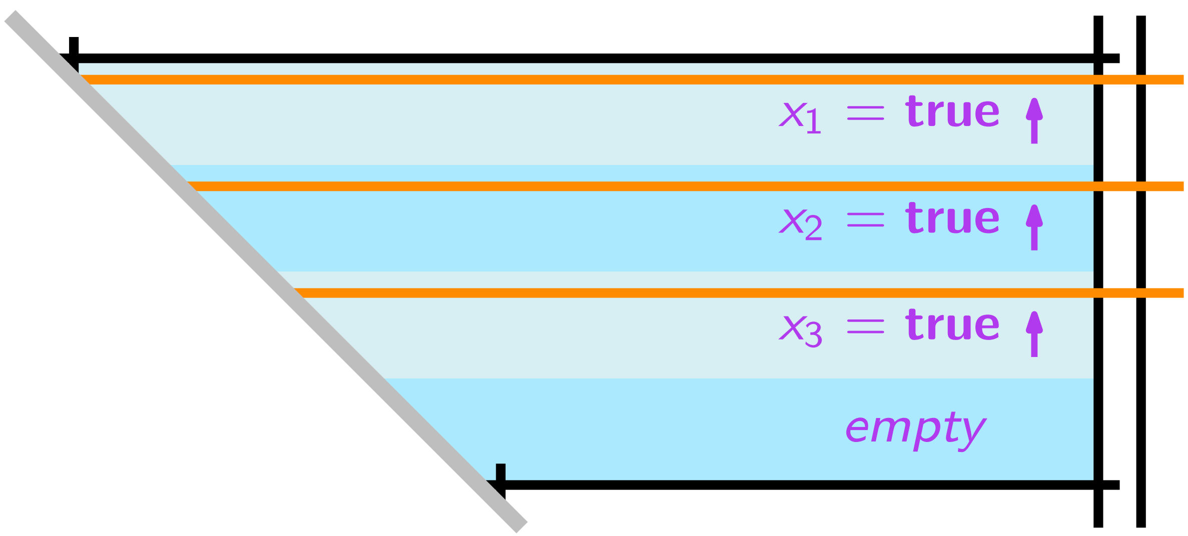

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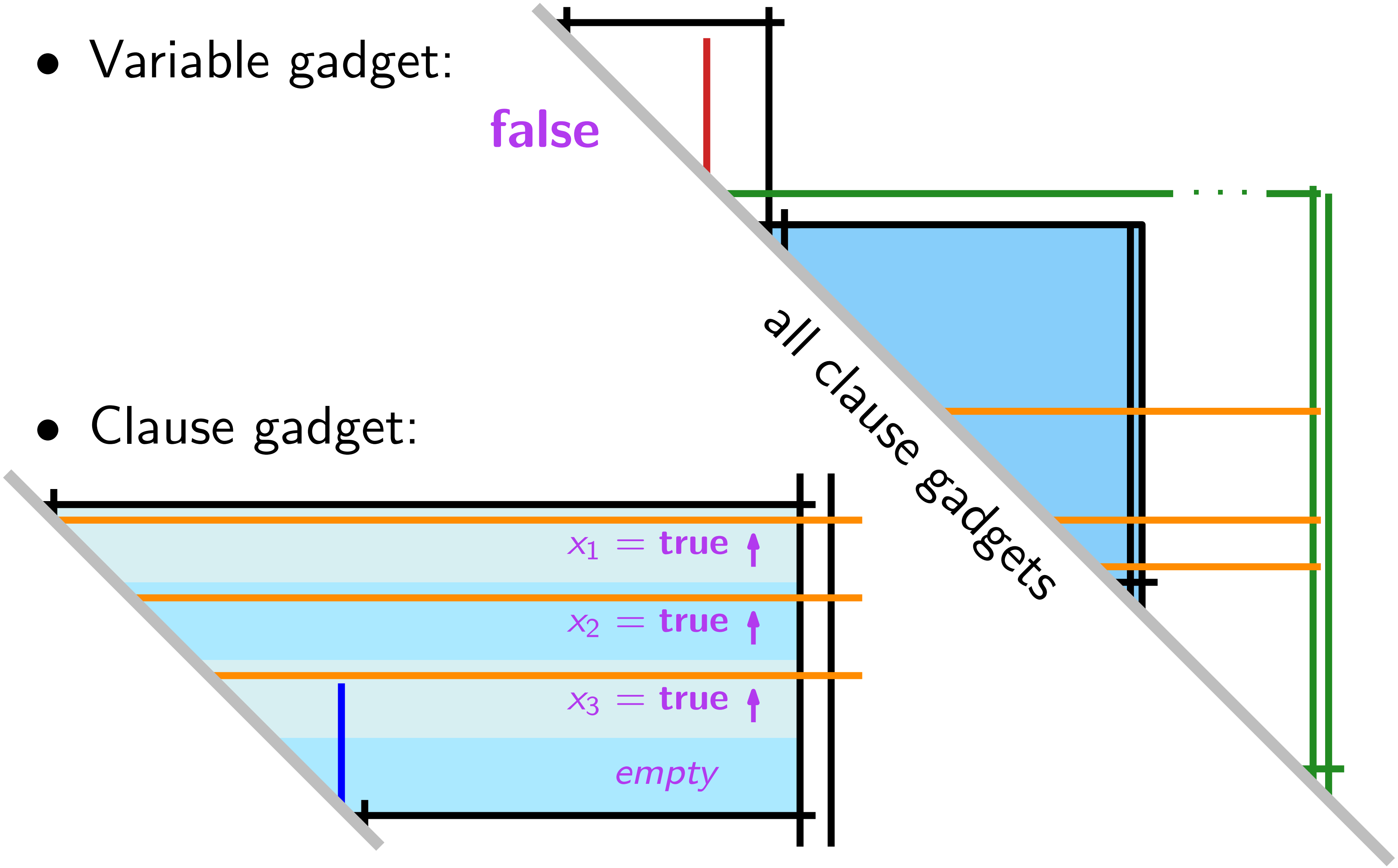

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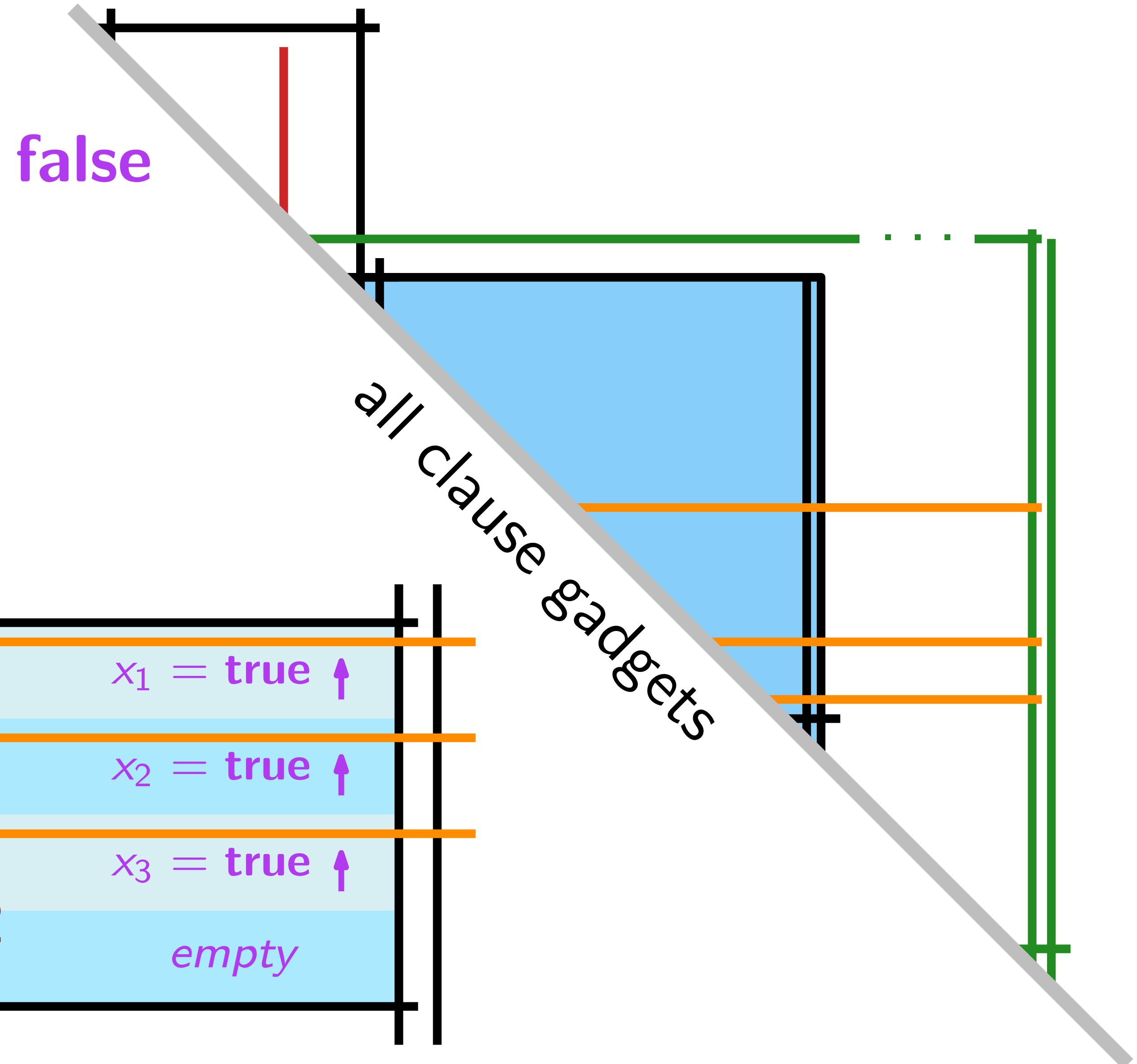

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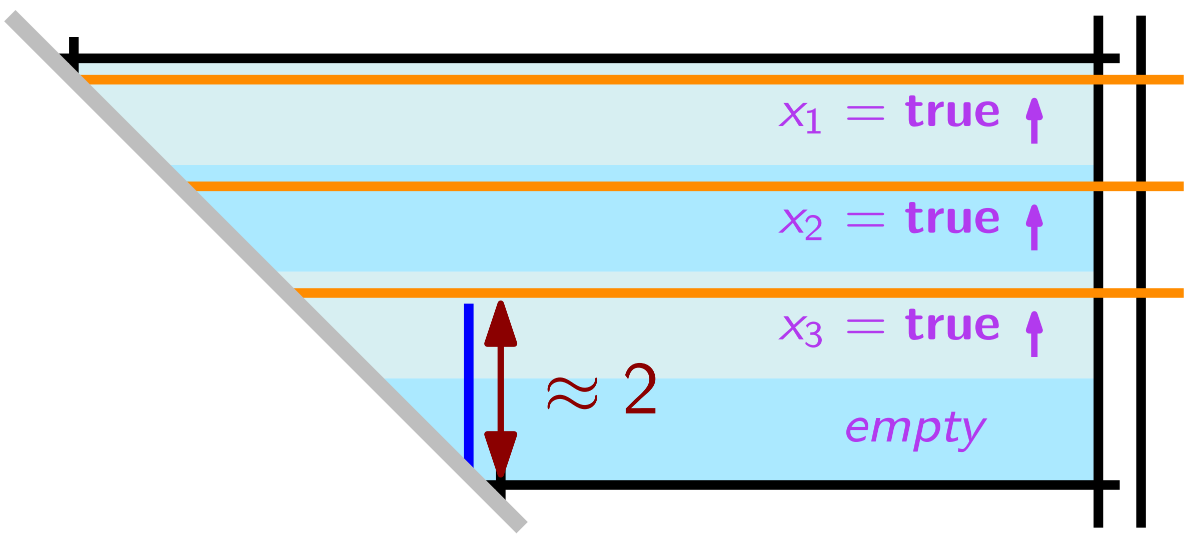

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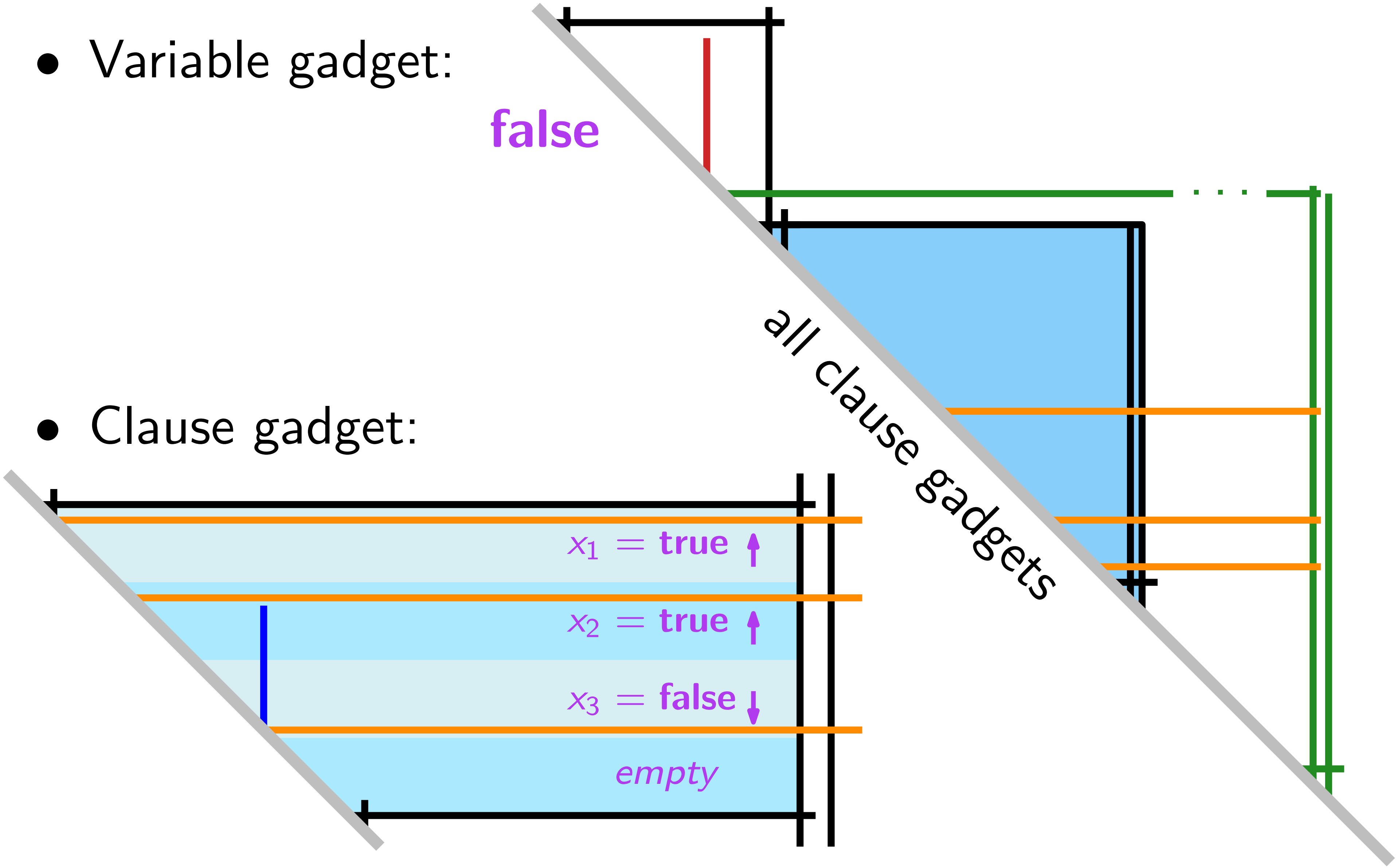

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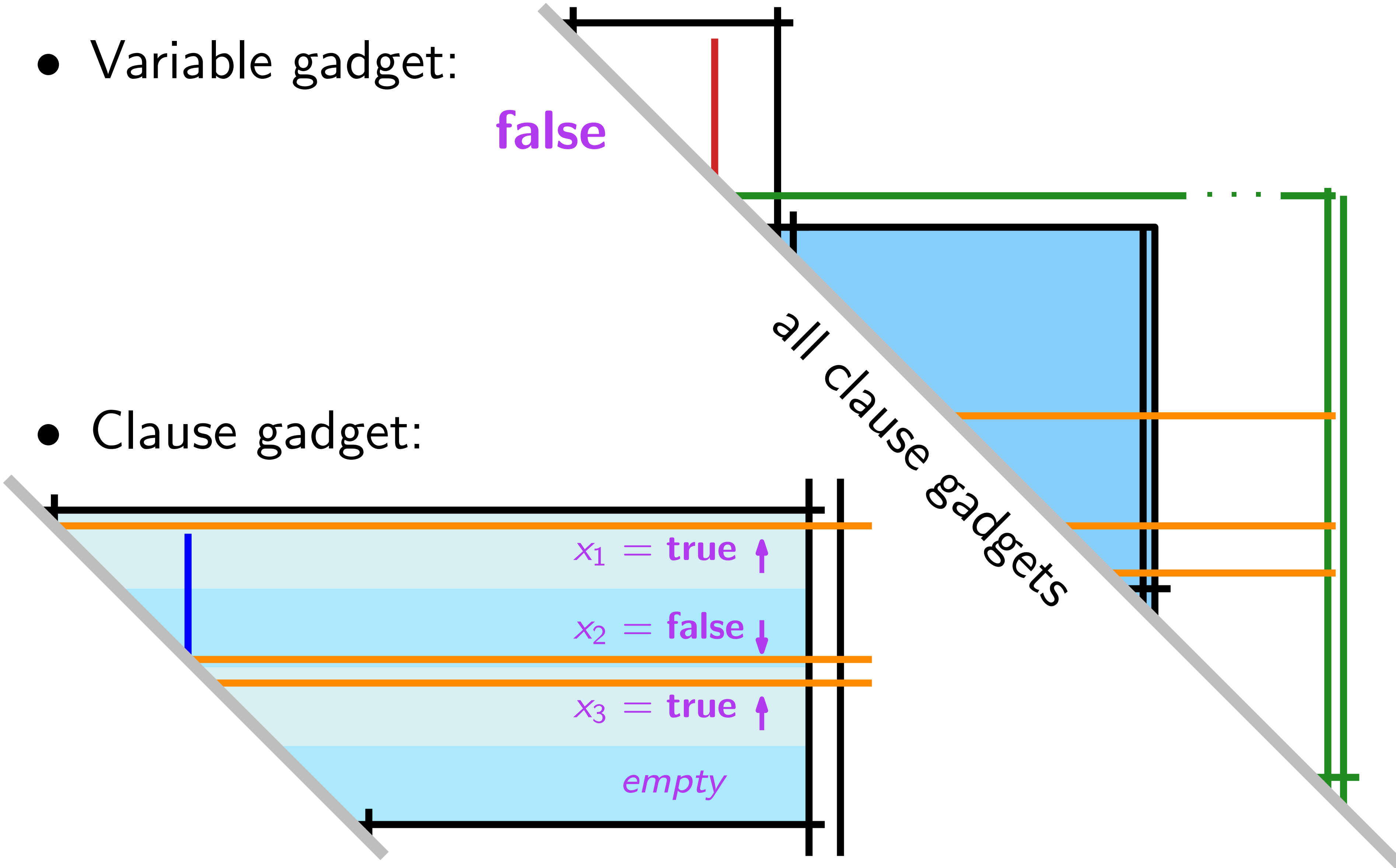

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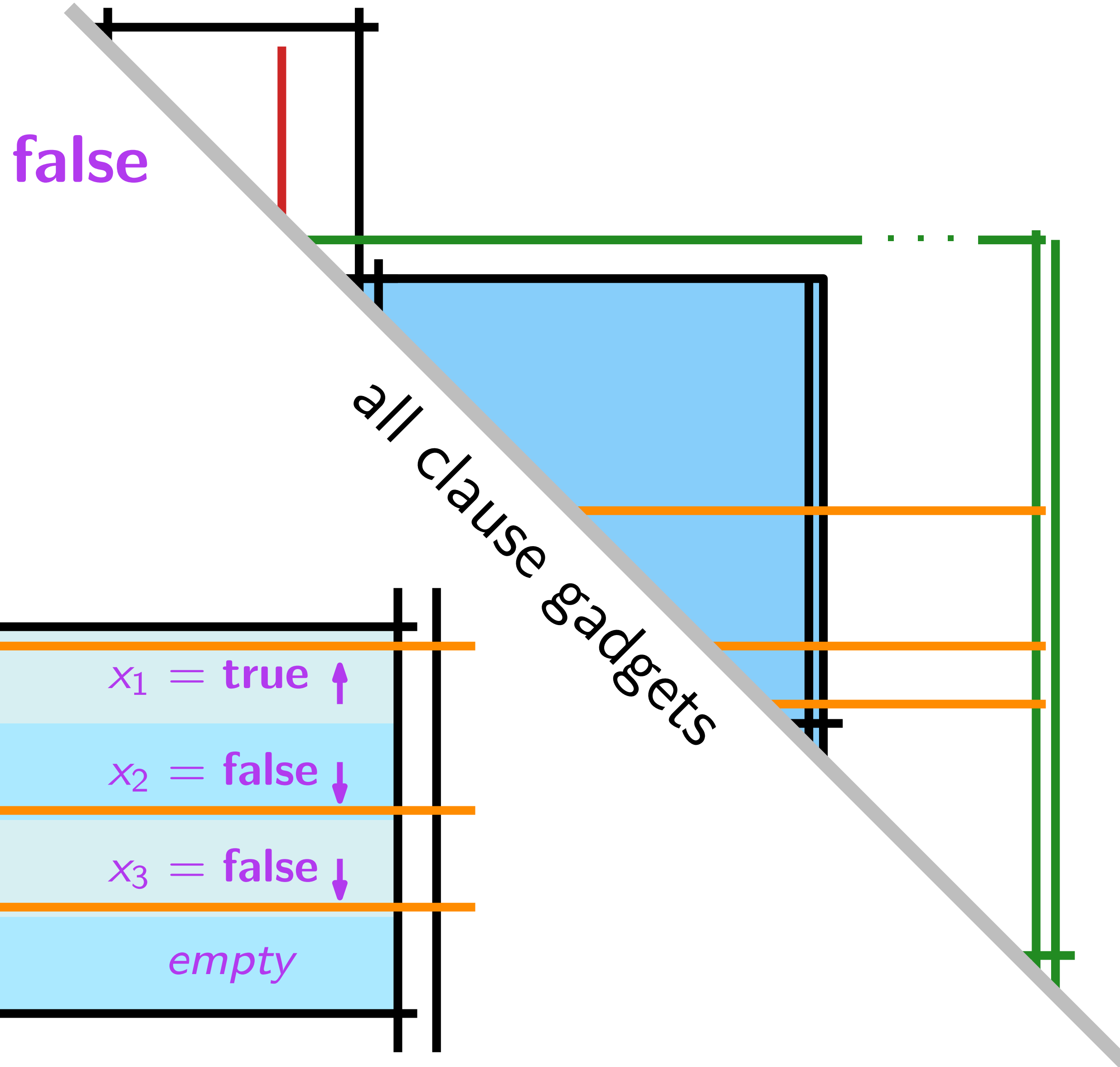

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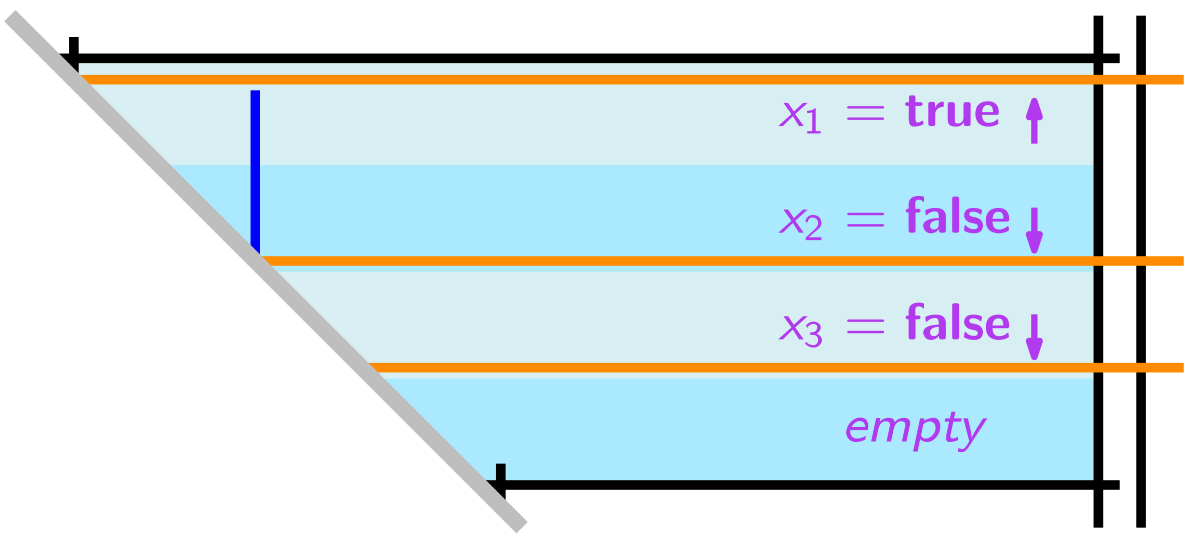

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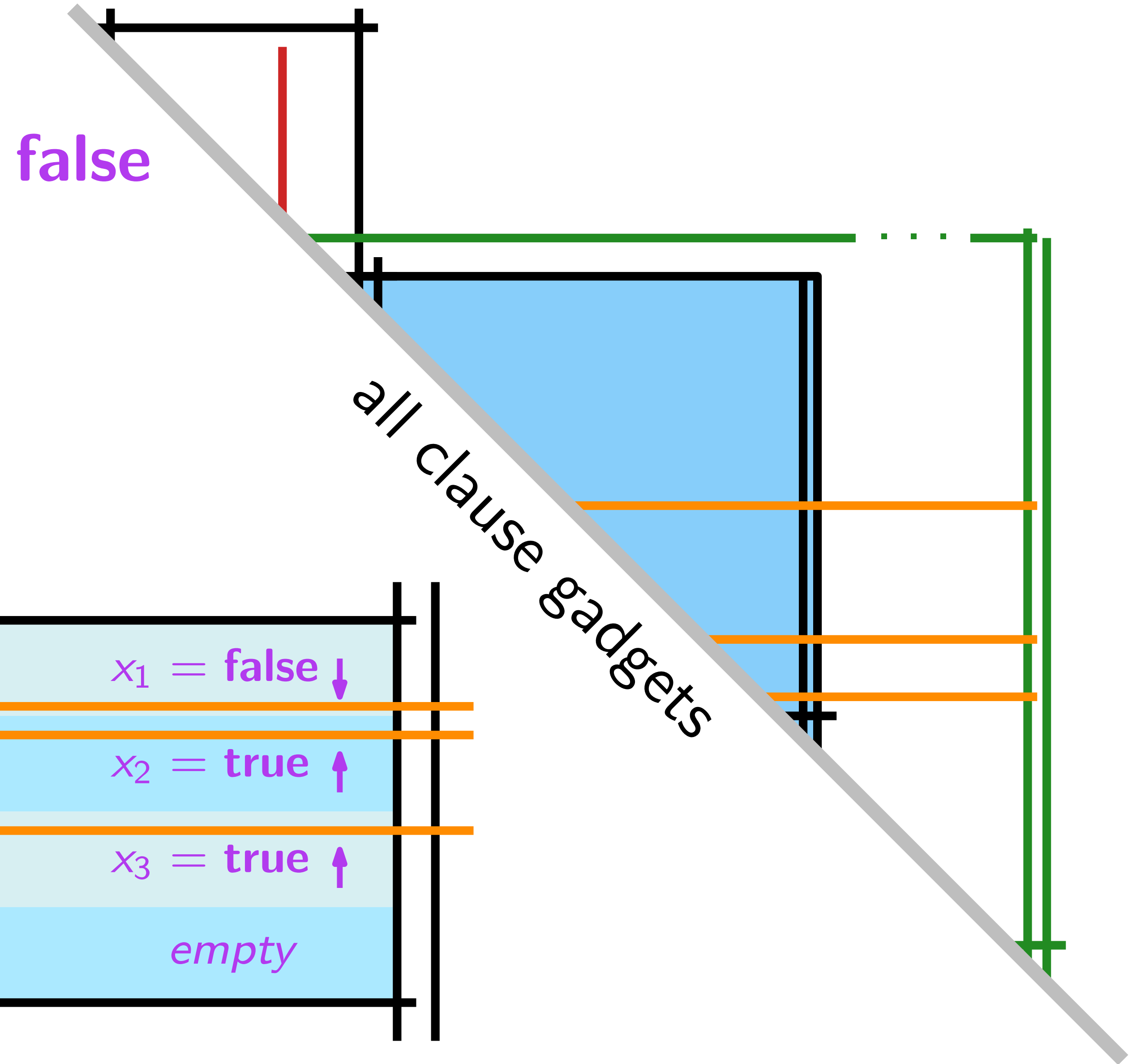

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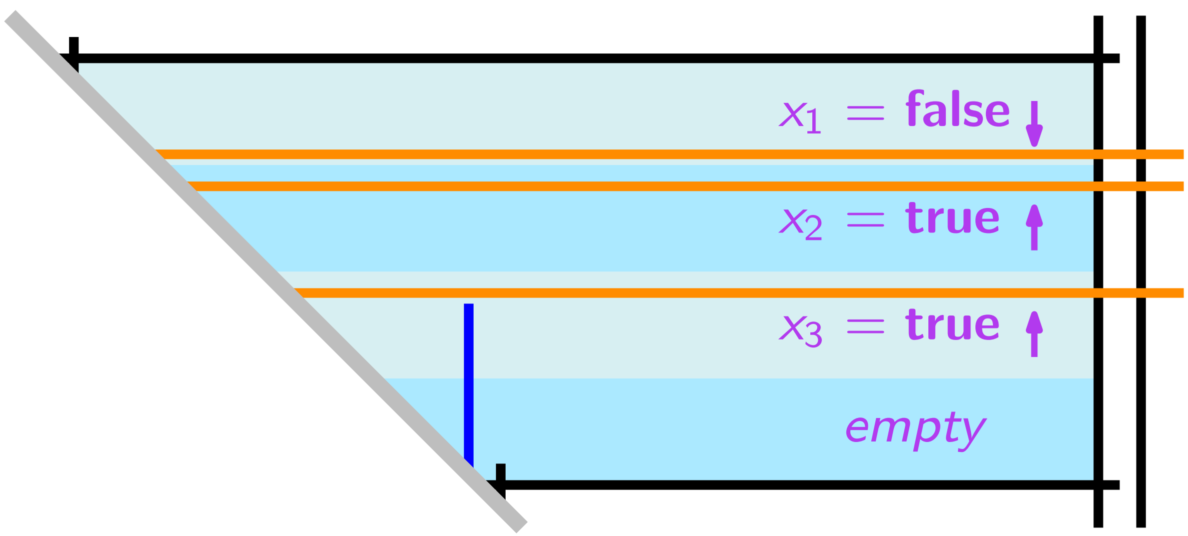

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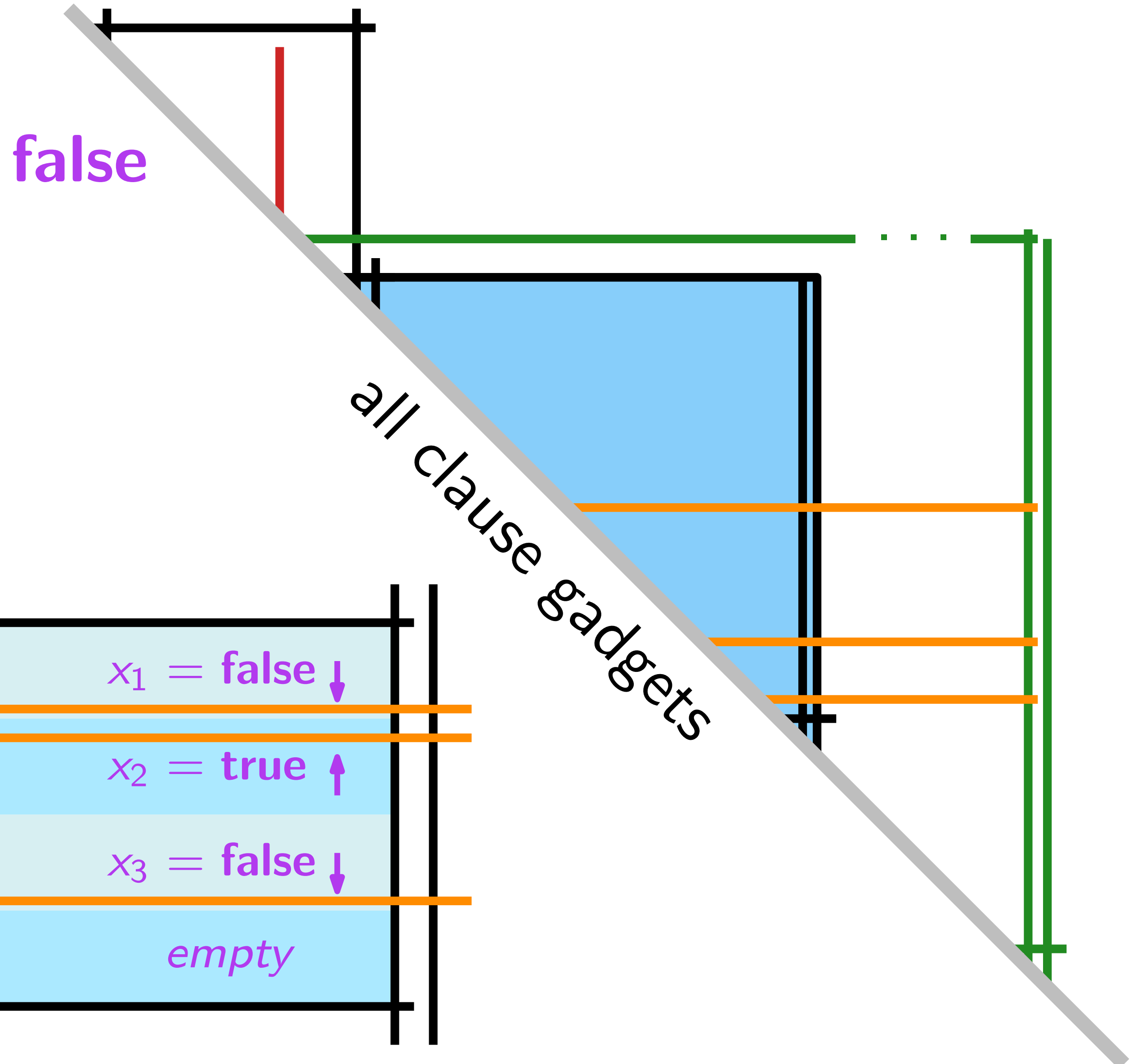

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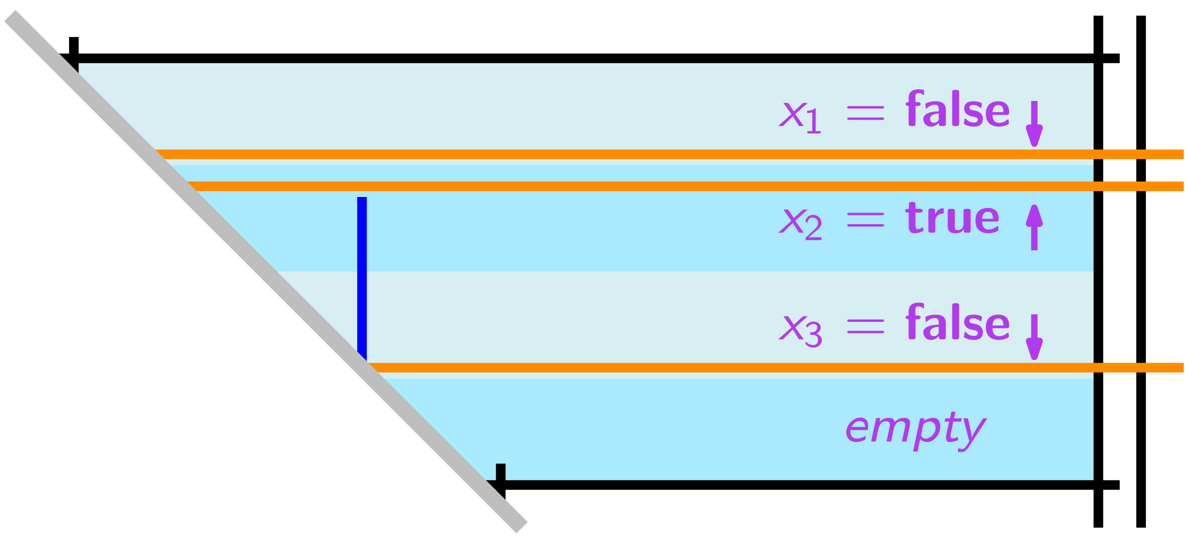

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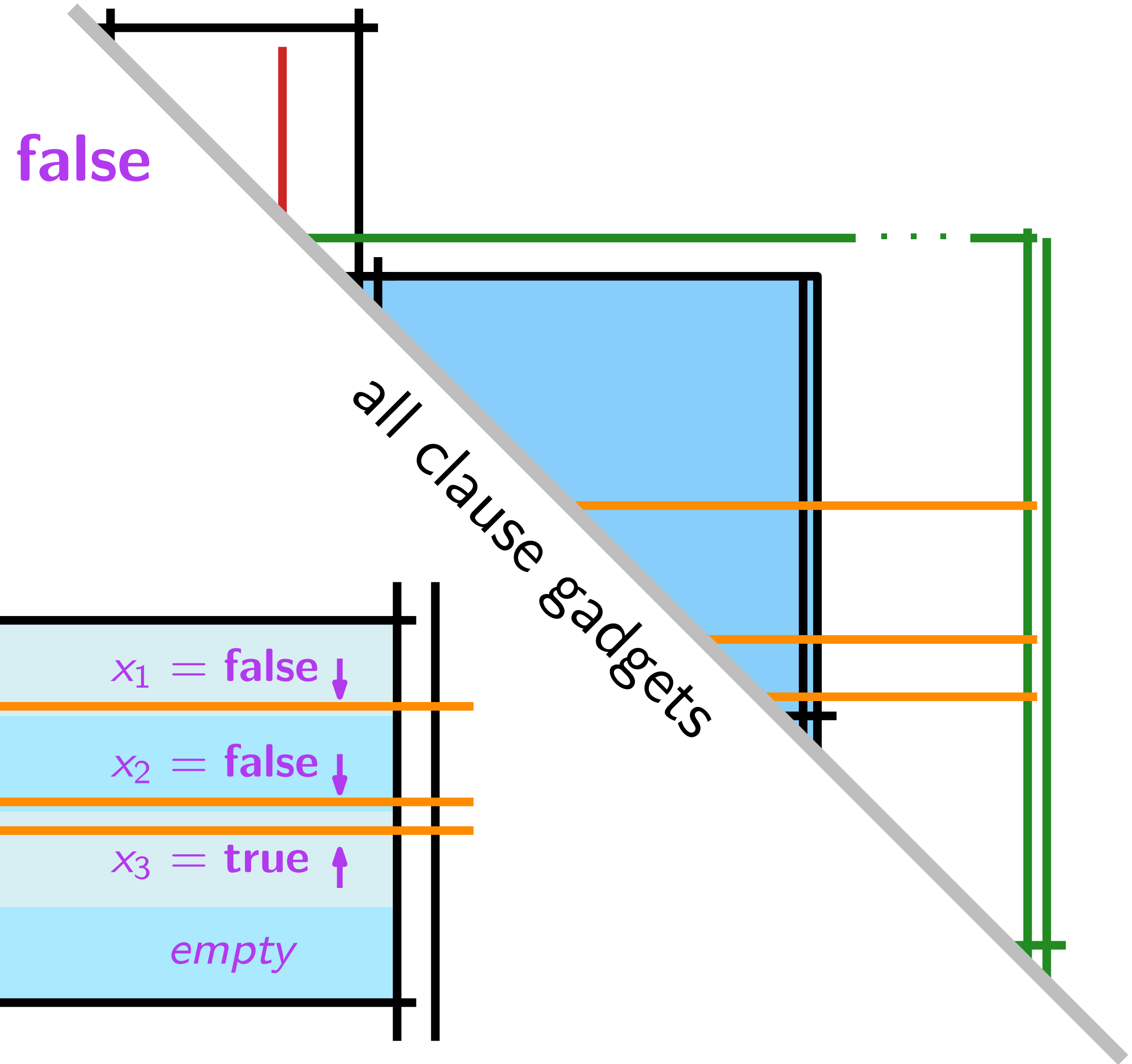

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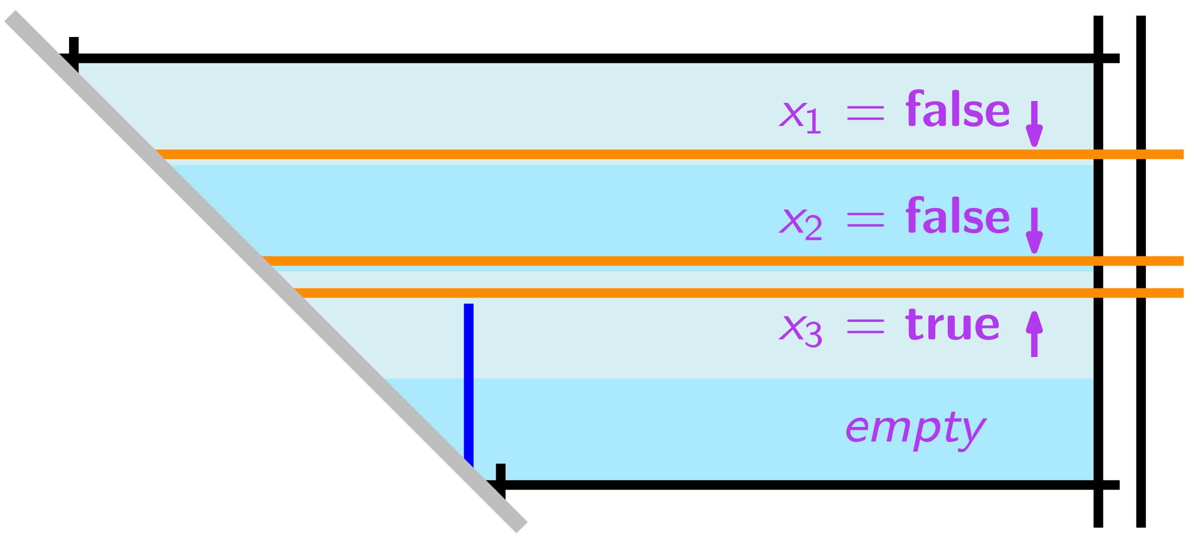

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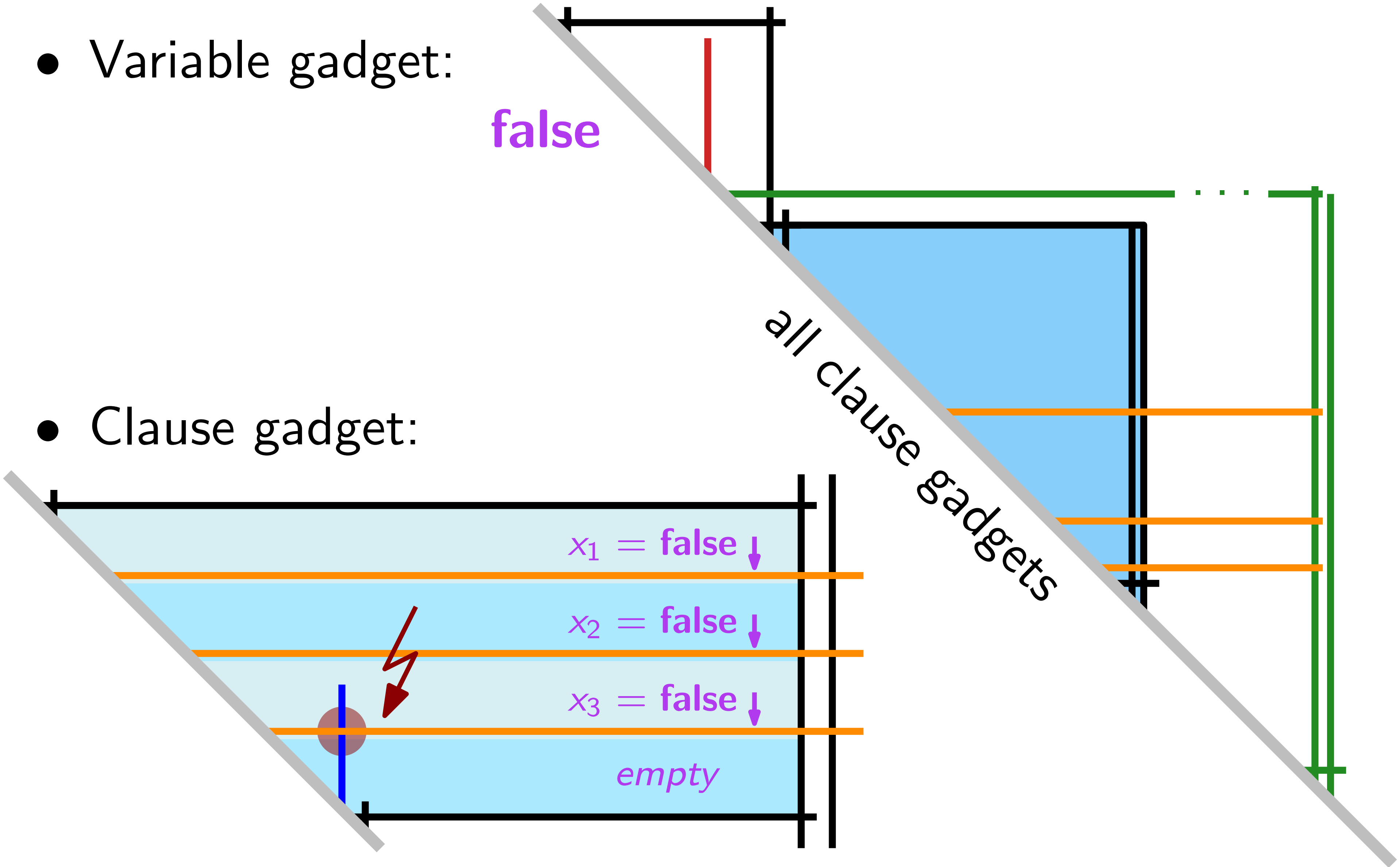

Example

MONOTONE-3-SAT formula:

$(x_1 \vee x_2 \vee x_3) \wedge$

$(x_2 \vee x_3 \vee x_4) \wedge$

$(\neg x_1 \vee \neg x_2 \vee \neg x_4)$

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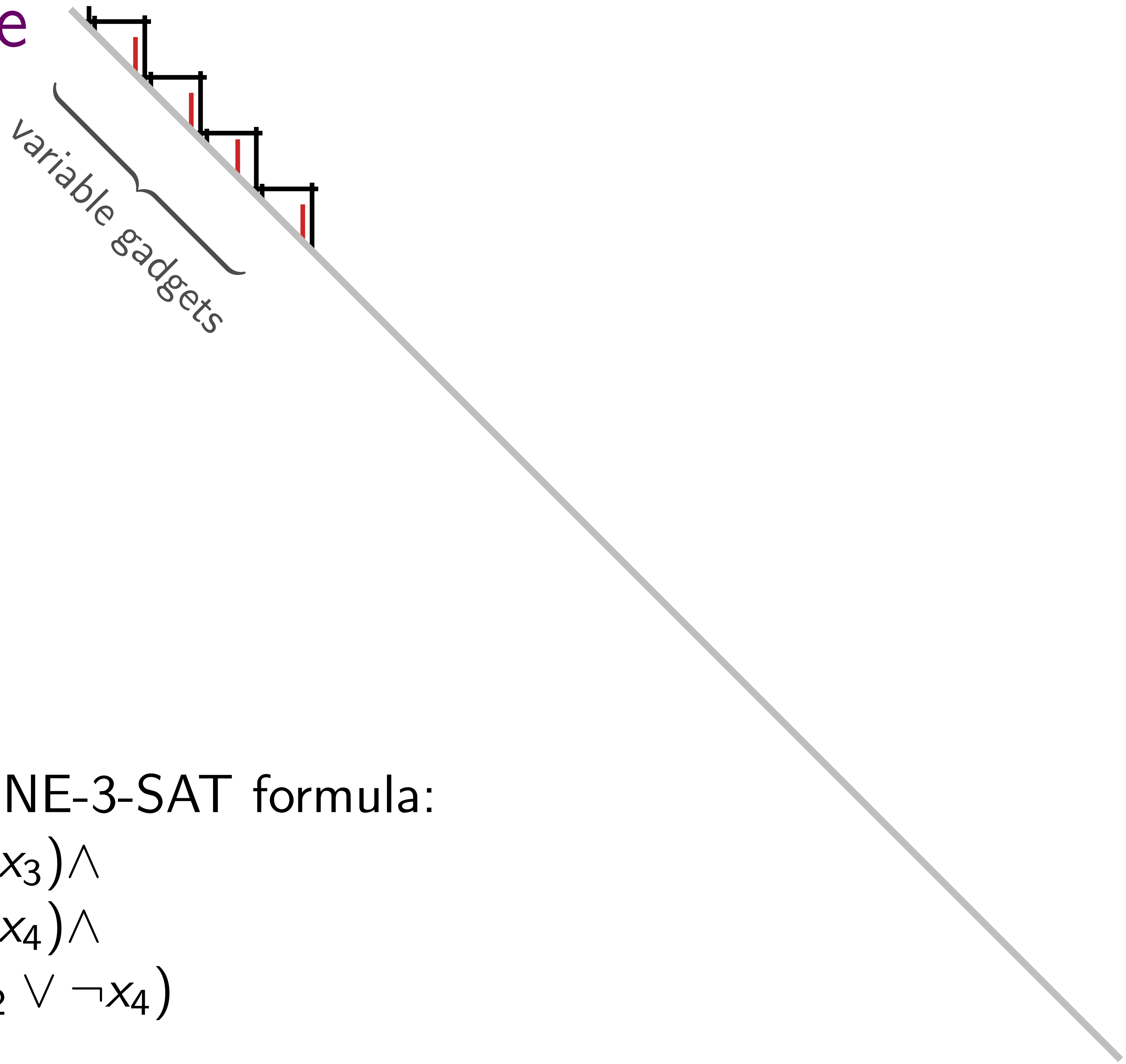

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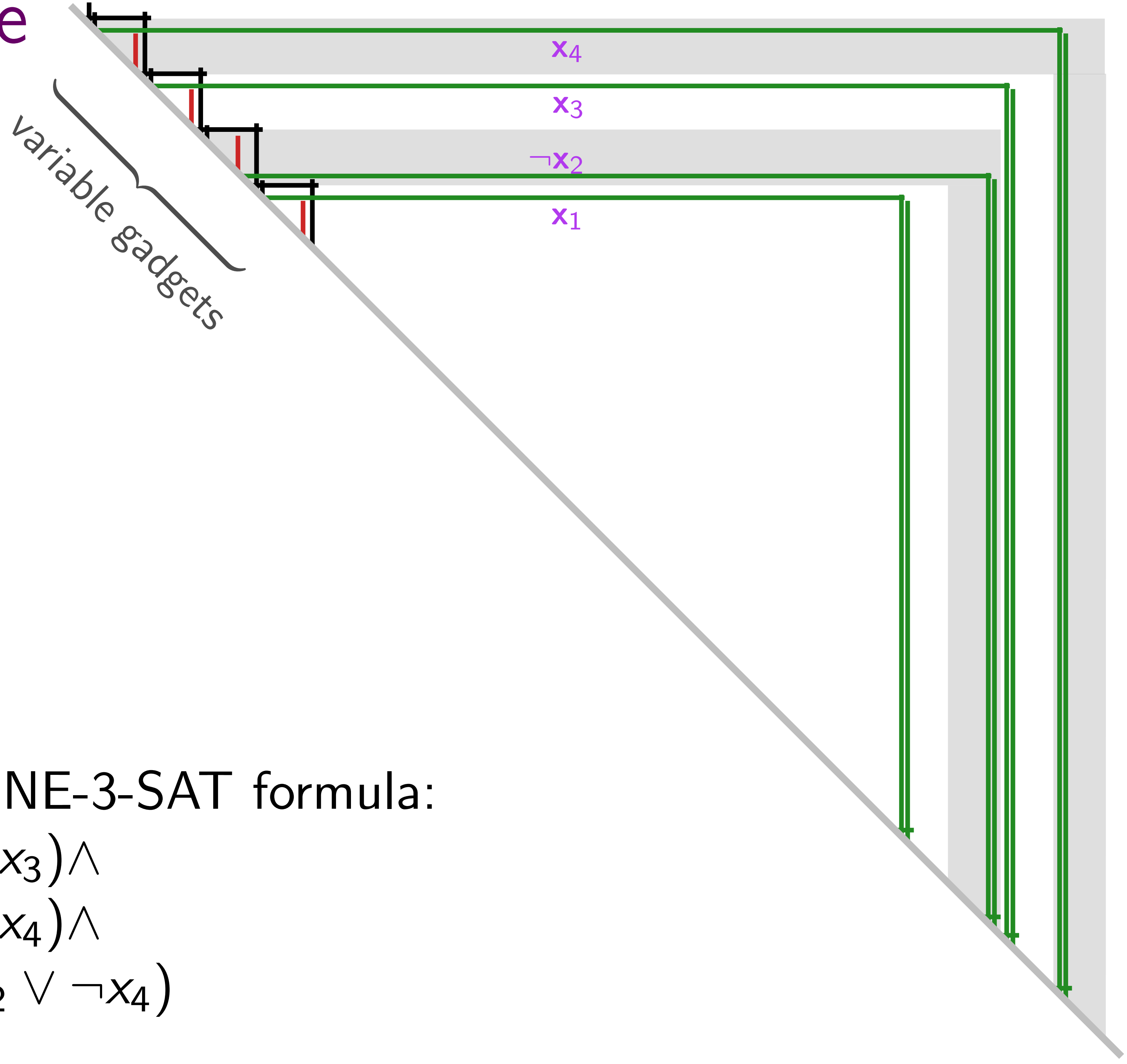

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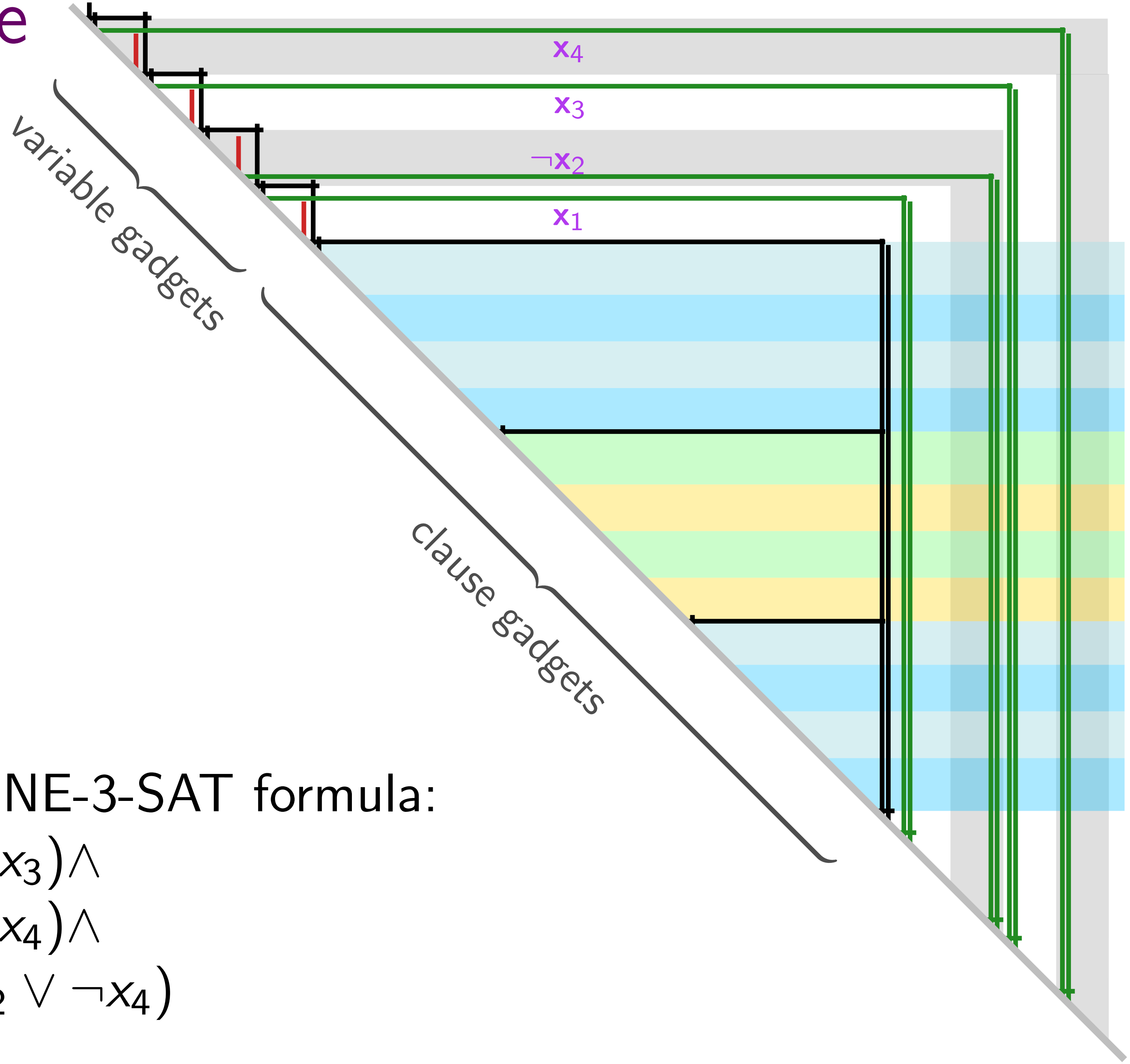

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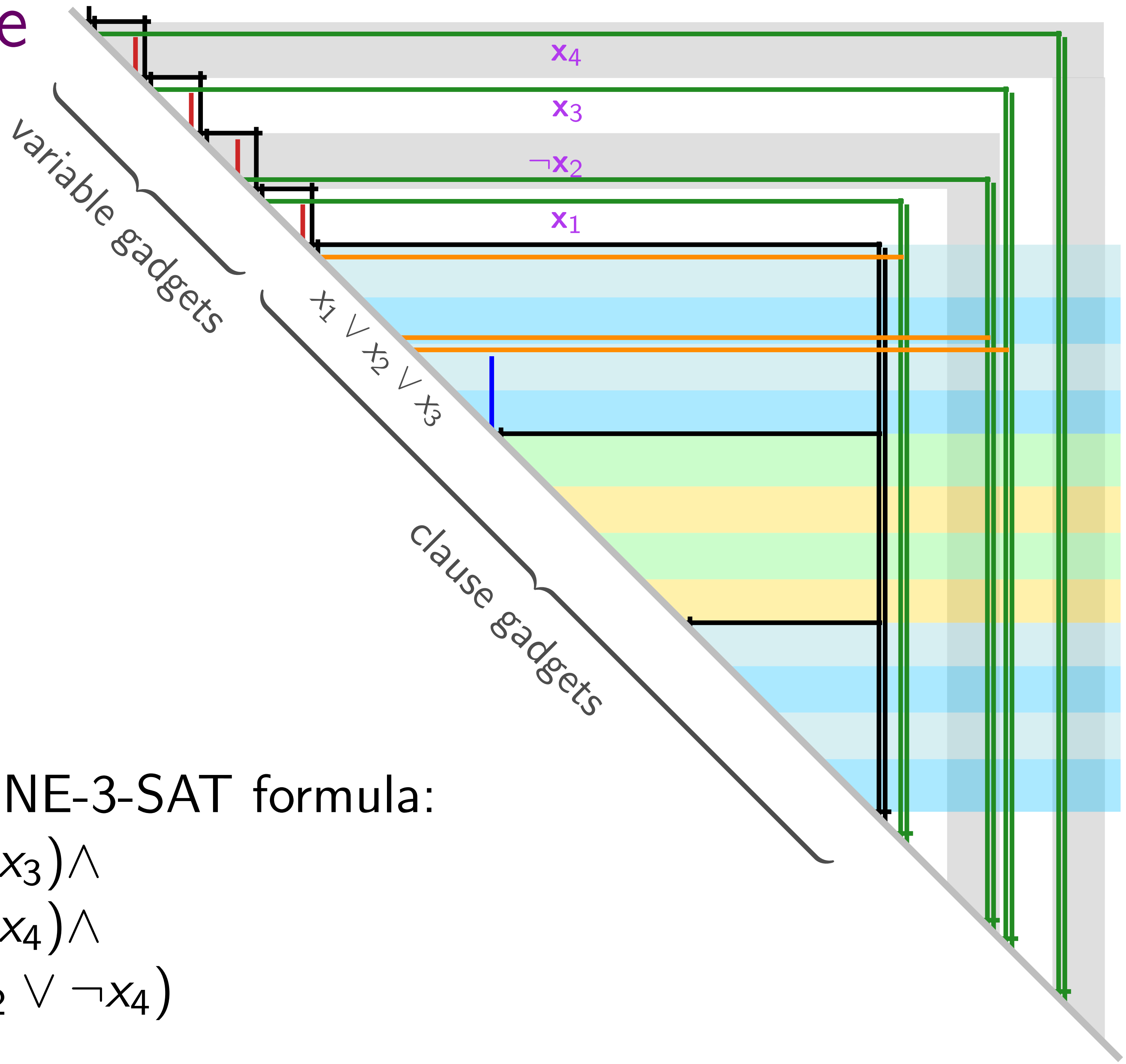

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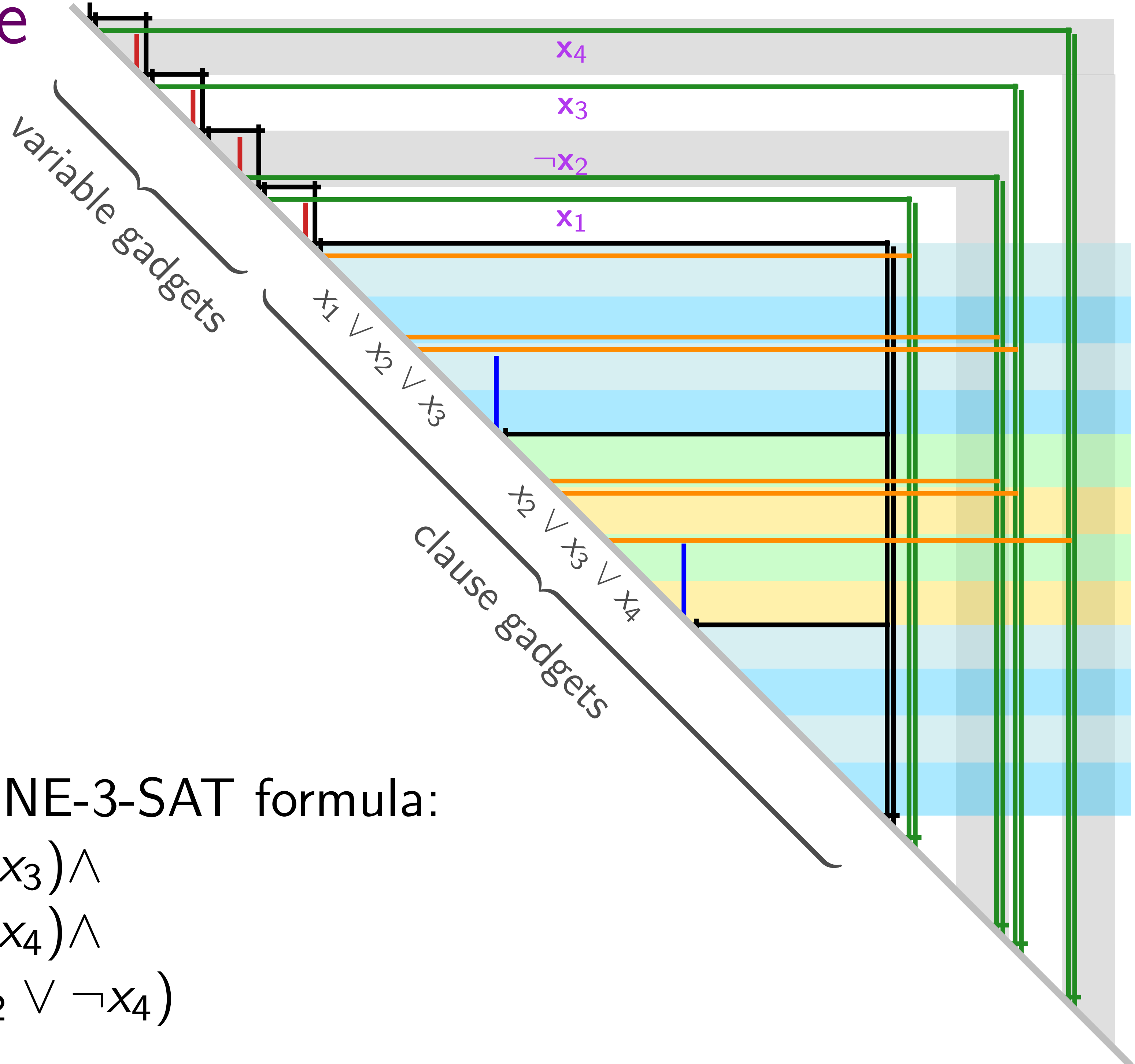

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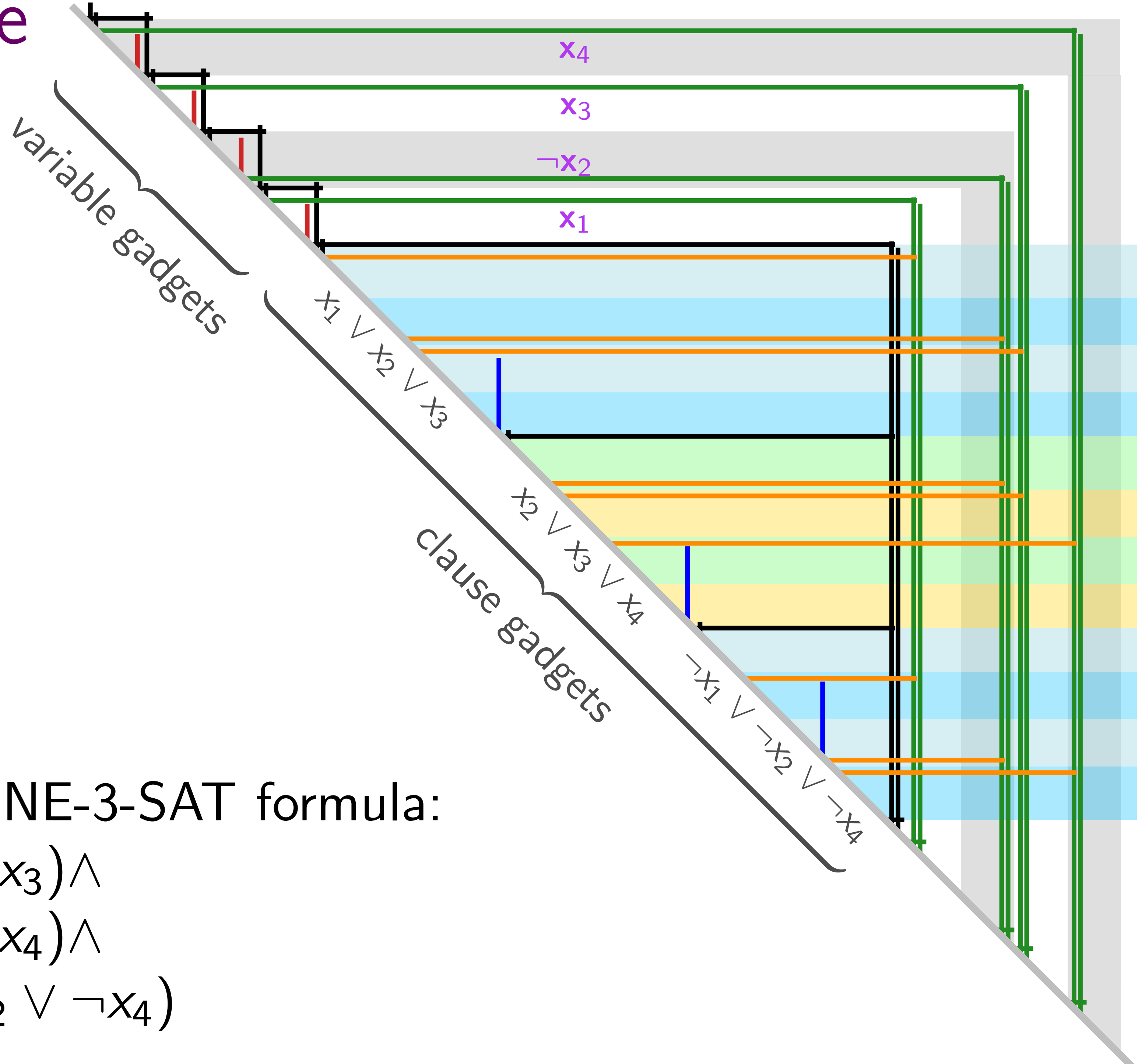

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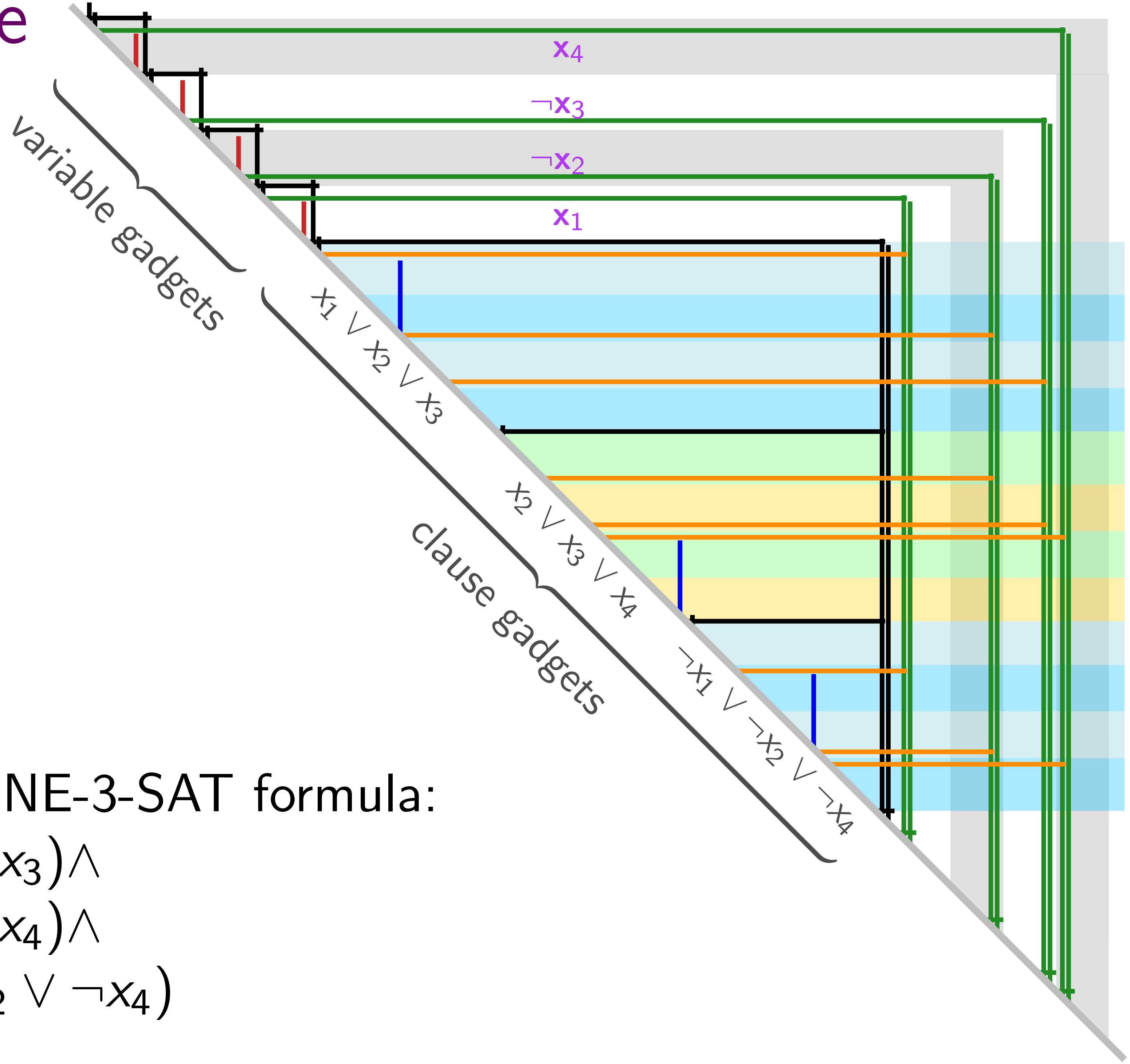

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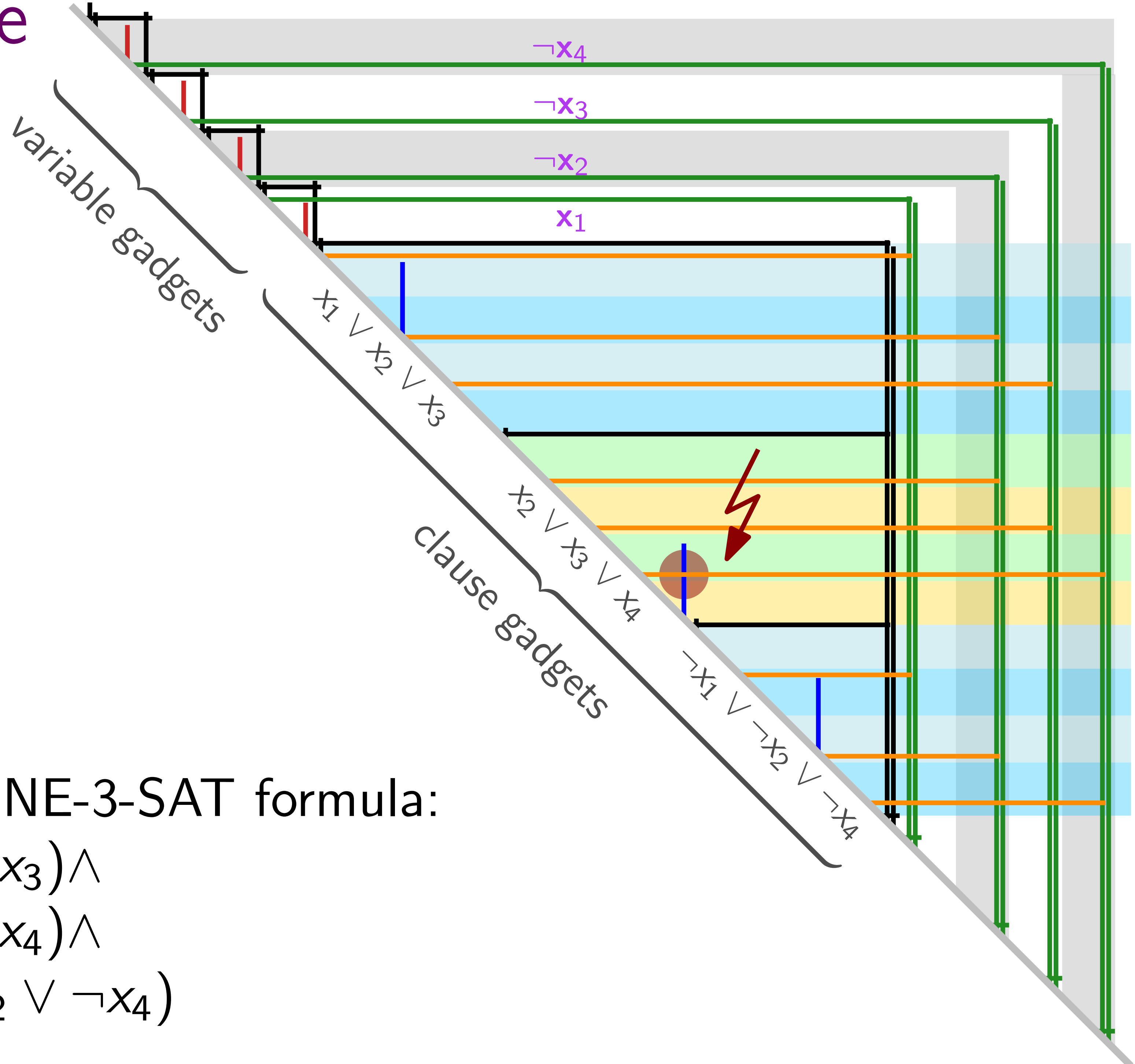

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$\text{STICK}^{\text{fix}}_{\text{AB}}$ without isolated vertices

$\star$	**$\text{STICK}_{\star}$**	**$\text{STICK}^{\text{fix}}_{\star}$**
	?	NP-complete
A	$O(\|A\|\|B\|)$	NP-complete
AB	$O(\|E\|)$	in general: NP-complete w/o isolated vtc.: $O((\|A\|+\|B\|)^2)$

Uniqueness Lemma

Lemma:

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Lemma: In all stick representations of an instance of $\mathsf{STICK}_{\mathsf{AB}}$, the order of vertices on the ground line is the same after removing all isolated vertices. This order can be found in time $O(|E|)$.

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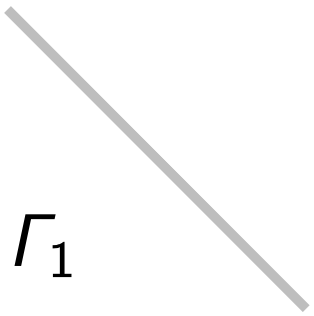

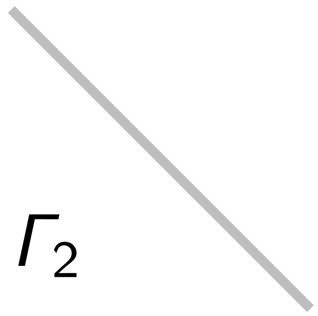

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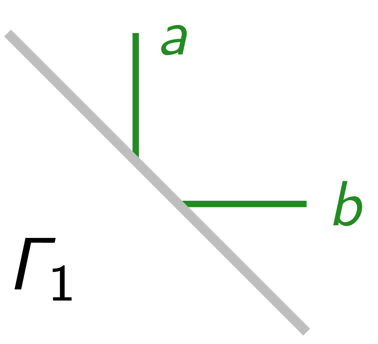

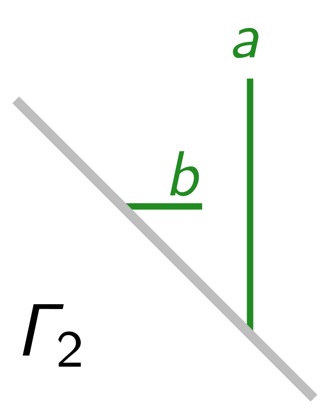

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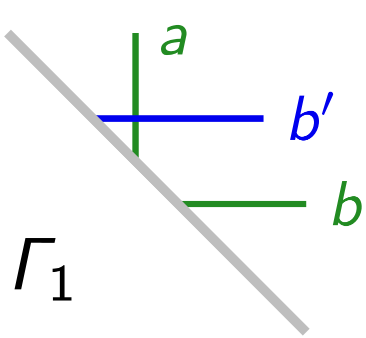

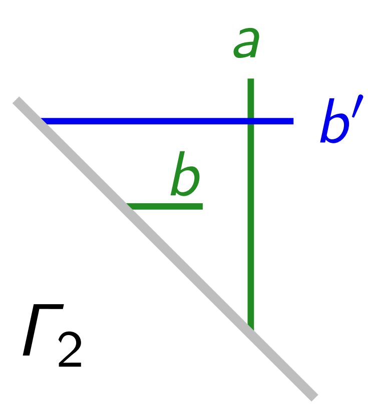

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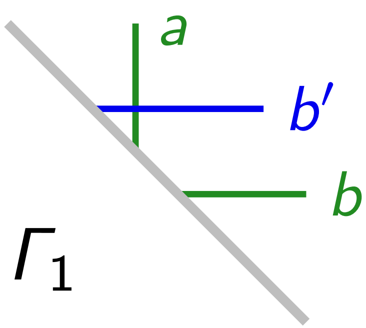

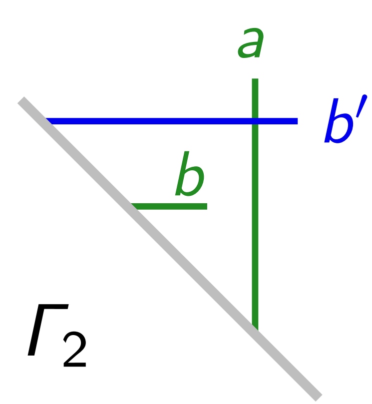

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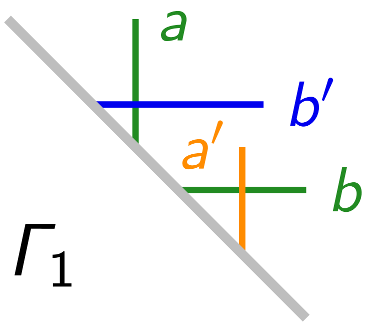

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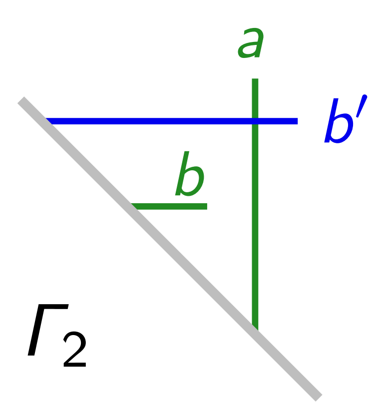

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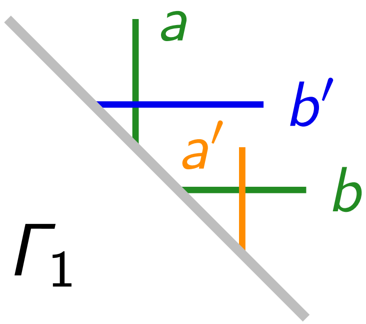

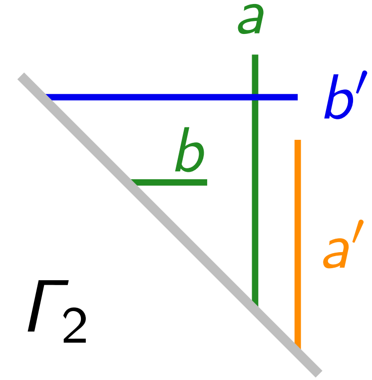

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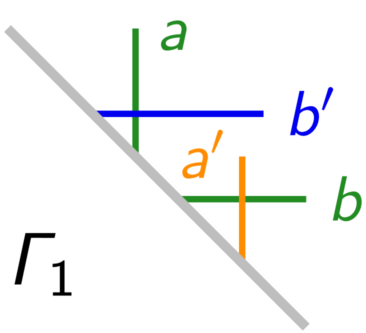

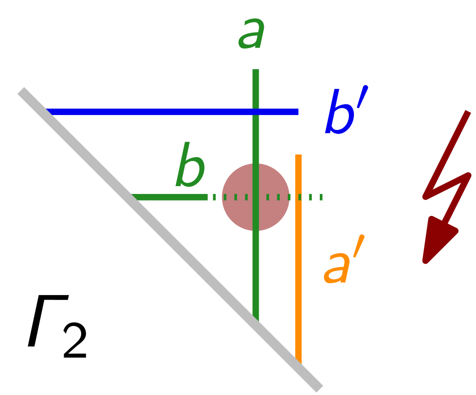

Linear Program for $\text{STICK}_{\text{AB}}^{\text{fix}}$

- No isolated vertices $\Rightarrow$ Compute ordering $v_1, \ldots, v_n$

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- No isolated vertices $\Rightarrow$ Compute ordering $v_1, \ldots, v_n$
- Variable x_i for the x-coordinate of v_i's footpoint
- Constraints: $x_1 < \cdots < x_n$
- For each vertex 2 more constraints incorporating the predefined stick-lengths:
  - for intersecting its last neighbor
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$\Rightarrow$ **Isolated vertices make $\text{STICK}_{\text{AB}}^{\text{fix}}$ NP-hard**

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