

Tight Bounds for Subgraph Isomorphism and Graph Homomorphism*

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Abstract

We prove that unless Exponential Time Hypothesis (ETH) fails, deciding if there is a homomorphism from graph G to graph H cannot be done in time $|V(H)|^{o(|V(G)|)}$. Combined with the reduction of Cygan, Pachocki, and Socala, our result rules out (subject to ETH) a possibility of $|V(G)|^{o(|V(G)|)}$ -time algorithm deciding if graph H is a subgraph of G . For both problems our lower bounds asymptotically match the running time of brute-force algorithms trying all possible mappings of one graph into another. Thus, our work closes the gap in the known complexity of these fundamental problems.

1 Introduction

A *homomorphism* $G \rightarrow H$ from an undirected graph G to an undirected graph H is a mapping from the vertex set $V(G)$ to $V(H)$ such that the image of every edge of G is an edge of H . Then the GRAPH HOMOMORPHISM problem $\text{HOM}(G, H)$ is the problem to decide for given graphs G and H , whether $G \rightarrow H$. GRAPH HOMOMORPHISM is a generic problem and many fundamental combinatorial problems like GRAPH COLORING and CLIQUE can be seen as its special cases. We refer to books of Hell and Nešetřil [11] and Lovász [14] for introduction to and applications of graph homomorphisms.

Solving $\text{HOM}(G, H)$ can be done by checking all possible mappings from an n -vertex graph G into an h -vertex graph H .¹ The running time of this brute-force algorithm is $\mathcal{O}(h^n) = 2^{\mathcal{O}(n \log h)}$. It was shown by Chen et al. [3] that under the ETH assumption, for any constant $\varepsilon > 0$, there is no $n^{o(k)}$ -time algorithm checking whether a given n -vertex graph contains a k -clique for any $k = \mathcal{O}(n^{1-\varepsilon})$. This implies, in particular, that GRAPH HOMOMORPHISM cannot be solved in time

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¹In order to obtain general results, throughout the paper we assume implicitly that $h = h(n)$ is a function of n . We assume that the function $h(n)$ is non-decreasing and time-constructible.

$2^{o(n \log h)}$ for h significantly larger than n (again, under the ETH assumption). At the same time this does not exclude the existence of a faster algorithm for some $h \leq n$. Moreover, one of the most natural special cases, the h -coloring problem (an n vertex graph can be colored in $h \leq n$ colors if and only if there is a homomorphism from the graph to an h -clique; for this reason, $\text{HOM}(G, H)$ is often called H -coloring of G), can be solved in time $2^n \text{poly}(n)$ as shown by Björklund et al. [2]. That is why the existence of an algorithm solving $\text{GRAPH HOMOMORPHISM}$ asymptotically faster than the brute-force was a major open problem in the area of Exact Exponential Algorithms [8, 15, 16, 17]. In [7], we have shown that unless ETH fails, there is no algorithm solving $\text{GRAPH HOMOMORPHISM}$ in time $2^{o\left(\frac{n \log h}{\log \log h}\right)}$ for every function $h(n)$. In this paper, we close the gap between the existing lower and upper bounds by ruling out a possibility of solving $\text{GRAPH HOMOMORPHISM}$ in time $2^{o(n \log h)}$ for every function $h(n)$.

Our result also implies a tight bound for the related $\text{SUBGRAPH ISOMORPHISM}$ problem. Here, for two given n -vertex graphs G and H , the task is to decide if G contains a subgraph isomorphic to H . As $\text{GRAPH HOMOMORPHISM}$, $\text{SUBGRAPH ISOMORPHISM}$ encompasses many fundamental problems including HAMILTONIAN CYCLE , BANDWIDTH , TRIANGLE PACKING , CLIQUE , and BI-CLIQUE . Again, a brute-force algorithm solves $\text{SUBGRAPH ISOMORPHISM}$ in time $2^{O(n \log n)}$ and a possibility of time $2^{o(n \log n)}$ solving $\text{SUBGRAPH ISOMORPHISM}$ was another long-standing open question in the area, see for example [1, 5, 6], and [9, Chapter 12]. Recently, Cygan, Pachoeki, and Socała [5] showed that $\text{GRAPH HOMOMORPHISM}$ can be solved by solving $2^{O(n)}$ instances of $\text{SUBGRAPH ISOMORPHISM}$. This reduction, together with [7], implied that $\text{SUBGRAPH ISOMORPHISM}$ cannot be solved in time $2^{o\left(\frac{n \log n}{\log \log n}\right)}$ unless ETH fails [5]. Combined with this reduction, our lower bound for homomorphisms rules out algorithms of running time $2^{o(n \log n)}$ for $\text{SUBGRAPH ISOMORPHISM}$, and closes the gap between upper and lower bounds for this problem as well.

We build the proof of our tight lower bounds for graph homomorphisms on [7]. As in [7], we obtain lower bounds for $\text{GRAPH HOMOMORPHISM}$ by reducing the 3-coloring problem on graphs of bounded degree to it. The crucial difference with [7], which allows us to obtain a tight bound, is that we reduce 3-coloring of graphs of degree d on n vertices to list homomorphism of $\frac{n}{r}$ -vertex graph to $\gamma(d)^r$ -vertex graph, where $\gamma(d)$ is a function that depends on d only. Then an $h^{o(n)}$ upper bound for n -vertex to h -vertex graph homomorphism would imply a subexponential $\gamma(4)^{o(n)}$ algorithm for 3-coloring on graphs of degree 4, contradicting ETH.

2 Preliminaries

2.1 Main Definitions

Let G be a graph, by $V(G)$ and $E(G)$ we denote the sets of vertices and edges of G , respectively. For a vertex $v \in V(G)$, by the neighborhood $N_G(v)$ we mean the set of all vertices of G adjacent to v . By the square of G we denote the graph G^2 , such that $V(G^2) = V(G)$, and $\{u, v\} \in E(G^2)$ if and only if there is a path of length at most two from u to v in G .

Let G be an n -vertex graph, $1 \leq k \leq n$ be an integer, and $V(G) = B_1 \sqcup B_2 \sqcup \dots \sqcup B_k$ be a partition of the set of vertices of G . Then a *grouping* of G with respect to the partition $V(G) = B_1 \sqcup B_2 \sqcup \dots \sqcup B_k$ is a graph \tilde{G} with vertices B_1, \dots, B_k such that B_i and B_j are adjacent in \tilde{G} if and only if there exist $u \in B_i$ and $v \in B_j$ such that $\{u, v\} \in E(G)$. To distinguish vertices of the graphs G and \tilde{G} , the vertices of \tilde{G} will be called *buckets*.

A *proper coloring* of a graph G is a function assigning a color to each vertex of G such that adjacent vertices have different colors. An *equitable coloring* is a proper coloring where the numbers of vertices of any two colors differ by at most one. An *injective coloring* is a proper coloring that assigns different colors to any pair of vertices that have a common neighbor (note that a proper coloring of the square of a graph G is an injective coloring of G).

For a positive integer k , we use $[k]$ to denote the set of integers $\{1, \dots, k\}$. All logarithms in this paper are logarithms to the base two.

2.2 Homomorphism and Subgraph Isomorphism

Let G and H be graphs. A mapping $\varphi : V(G) \rightarrow V(H)$ is a *homomorphism* if for every edge $\{u, v\} \in E(G)$ its image $\{\varphi(u), \varphi(v)\} \in E(H)$. If there exists a homomorphism from G to H , we write $G \rightarrow H$. The GRAPH HOMOMORPHISM problem $\text{HOM}(G, H)$ asks whether or not $G \rightarrow H$. We also use the following generalization of graph homomorphism. Assume that for each vertex v of G there is an assigned list $\mathcal{L}(v) \subseteq V(H)$ of vertices. A *list homomorphism* of G to H , also known as a list H -colouring of G , with respect to the lists \mathcal{L} , is a homomorphism $\varphi : V(G) \rightarrow V(H)$, such that $\varphi(v) \in \mathcal{L}(v)$ for all $v \in V(G)$. Then the LIST GRAPH HOMOMORPHISM problem $\text{LIST-HOM}(G, H)$ asks whether or not graph G with lists \mathcal{L} admits a list homomorphism to H with respect to \mathcal{L} .

In the SUBGRAPH ISOMORPHISM problem one is given two graphs G and H and the question is whether G contains a subgraph isomorphic to H .

2.3 Exponential Time Hypothesis

Our lower bound is based on the well-known complexity hypothesis of Impagliazzo, Paturi, and Zane [12], see [4, 13] for an overview of the hypothesis and its implications.

Exponential Time Hypothesis (ETH): There is a constant $s > 0$ such that 3-CNF-SAT with n variables and m clauses cannot be solved in time $2^{sn}(n+m)^{\mathcal{O}(1)}$.

Let us remind that in the 3-COLORING problem the task is to decide whether a given graph admits a proper coloring in three colors. We will need the following folklore lemma. It follows from the fact that (unless ETH fails) 3-COLORING on graphs of average degree four cannot be solved in subexponential time, see e.g. Theorem 3.2 in [13], and the classical reduction, see e.g. [10], for 3-COLORING on degree-four graphs.

Lemma 1. *Unless ETH fails, there exists a constant $q > 0$ such that there is no algorithm solving 3-COLORING on n -vertex graphs of maximum degree four in time $\mathcal{O}(2^{qn})$.*

3 Auxiliary Lemmata

In this section we provide auxiliary lemmata about colorings which will be used to prove lower bounds for GRAPH HOMOMORPHISM and SUBGRAPH ISOMORPHISM.

3.1 Balanced Colorings

In the following we show how to construct a specific “balanced” coloring of a graph in polynomial time. Let G be a graph of constant maximum degree. The coloring of G we want to construct

should satisfy three properties. First, it should be a proper coloring of G^2 . Then the size of each color class should be bounded as well as the number of edges between vertices from different color classes. More precisely.

Lemma 2. *For any constant d , there exist constants $\alpha, \beta, \tau > 1$ and a polynomial time algorithm that for a given graph G on n vertices of maximum degree d and an integer $\tau \leq L \leq \frac{n(d^2-1)}{2d^2(d^2+1)}$, finds a coloring $c: V(G) \rightarrow [L]$ satisfying the following properties:*

1. *The coloring c is a proper coloring of G^2 .*
2. *There are only a few vertices of each color: for all $i \in [L]$,*

$$|c^{-1}(i)| \leq \left\lceil \alpha \cdot \frac{n}{L} \right\rceil. \quad (1)$$

3. *There are only a few edges of G between each pair of colors: For all $i \neq j \in [L]$, we have*

$$k_{i,j} := |\{\{u, v\} \in E(G) : c(u) = i, c(v) = j\}| \leq K_{i,j} := \left\lceil \beta \cdot \frac{\min\{|c^{-1}(i)|, |c^{-1}(j)|\}}{L} \right\rceil.$$

Proof. The algorithm starts by constructing greedily an independent set I of G^2 of size $\left\lfloor \frac{n}{d^2+1} \right\rfloor$. Since the maximum vertex degree of G^2 does not exceed d^2 , this is always possible. We construct a partial coloring of G^2 by coloring the vertices of I in L colors such that the obtained coloring is an equitable coloring of $G^2[I]$. Since I is an independent set in G^2 , such a coloring can be easily constructed in polynomial time. In the obtained partial equitable coloring, we have that for every $i \in [L]$

$$|c^{-1}(i)| \geq \left\lfloor \frac{n}{L(d^2+1)} \right\rfloor \geq \frac{n}{2Ld^2} \quad (2)$$

(recall that $L \leq \frac{n(d^2-1)}{2d^2(d^2+1)}$). Let us note that the obtained precoloring of G^2 clearly satisfies the first and the third conditions of the lemma. Since the size of every $c^{-1}(i)$, $i \in [L]$, does not exceed $|c^{-1}(i)| \leq \left\lceil \frac{n}{L} \right\rceil$, the second condition of the lemma also holds for every $\alpha > 1$.

We extend the precoloring of G^2 to the required coloring by the following greedy procedure: We select an uncolored vertex v and color it by a color from $[L]$ such that the new partial coloring also satisfies the three conditions of the lemma. In what follows, we prove that such a greedy choice of a color is always possible.

Coloring of a vertex v with a color i can be forbidden only because it breaks one of the three conditions. Let us count, how many colors can be forbidden for v by each of the three constraints.

1. Vertex v has at most d^2 neighbors in G^2 , so the first constraint forbids at most d^2 colors.
2. The second constraint forbids all the colors that are “fully packed” already. The number of such colors is at most $\frac{n}{\frac{n}{L}} = \frac{L}{\alpha}$.
3. To estimate the number of colors forbidden by the third condition, we go through all the neighbors of v . A neighbor $u \in N_G(v)$ forbids a color i if coloring v by i exceeds the allowed bound on $k_{i,c(u)}$. Hence to estimate the number of such forbidden colors i (for every fixed

vertex u) we need to estimate how many values of $k_{i,c(u)}$ can reach the allowed upper bound $K_{i,c(u)}$. We have that

$$\begin{aligned} |\{i: k_{i,c(u)} = K_{i,c(u)}\}| &\stackrel{\text{by (2)}}{\leq} \left| \left\{ i: k_{i,c(u)} \geq \frac{\beta n}{2L^2 d^2} \right\} \right| = \left| \left\{ i: k_{i,c(u)} \cdot \frac{2L^2 d^2}{\beta n} \geq 1 \right\} \right| \\ &\leq \sum_{i \in [L]} k_{i,c(u)} \cdot \frac{2L^2 d^2}{\beta n}. \end{aligned}$$

The number of edges between vertices of the same color $c(u)$ and all other vertices of the graph does not exceed the cardinality of the color class $c(u)$ times d . Thus we have

$$\begin{aligned} \sum_{i \in [L]} k_{i,c(u)} \cdot \frac{2L^2 d^2}{\beta n} &\leq d |c^{-1}(c(u))| \cdot \frac{2L^2 d^2}{\beta n} \stackrel{\text{by (1)}}{\leq} d \left\lceil \frac{\alpha n}{L} \right\rceil \cdot \frac{2L^2 d^2}{\beta n} \\ &\leq d \frac{2\alpha n}{L} \cdot \frac{2L^2 d^2}{\beta n} = \frac{4\alpha L d^3}{\beta}. \end{aligned}$$

Therefore,

$$|\{i: k_{i,c(u)} = K_{i,c(u)}\}| \leq \frac{4\alpha L d^3}{\beta}.$$

Since the degree of v in G does not exceed d , we have that the number of colors forbidden by the third constraint is at most $\frac{4\alpha L d^4}{\beta}$.

Thus, the total number of colors forbidden by all the three constraints for the vertex v is at most

$$d^2 + \frac{L}{\alpha} + \frac{4\alpha L d^4}{\beta}.$$

By taking sufficiently large constants α , β , and τ , say $\alpha = 4$, $\beta = 16\alpha^2 d^4$, and $\tau = 2d^2 + 1$, we guarantee that this expression is less than $L - 1$ for every $L \geq \tau$. Therefore, there always exists a vacant color for the vertex v which concludes the proof. \square

Now with a help of Lemma 2, we describe a way to construct a specific grouping of a graph. The properties of such groupings are crucial for the final reduction.

Lemma 3. *For any constant d , there exists a constant $\lambda = \lambda(d)$ and a polynomial time algorithm that for a given graph G on n vertices of maximum degree d and an integer $r \leq \sqrt{\frac{n}{2\lambda}}$, finds a grouping \tilde{G} of G and a coloring $\tilde{c}: V(\tilde{G}) \rightarrow [\lambda r]$ such that*

1. *The number of buckets of \tilde{G} is*

$$|V(\tilde{G})| \leq \frac{|V(G)|}{r};$$

2. *The coloring \tilde{c} is a proper coloring of \tilde{G}^2 ;*

3. *Each bucket $B \in V(\tilde{G})$ is an independent set in G , i.e. for every $u, v \in B$, $\{u, v\} \notin E(G)$;*

4. For every pair of buckets $B_1, B_2 \in V(\tilde{G})$ there is at most one edge between them in G , i.e.

$$|\{\{u, v\} \in E(G) : u \in B_1, v \in B_2\}| \leq 1.$$

Proof. Let $\beta = \beta(d)$ be a constant provided by Lemma 2 and let $L = \lambda r$ for $\lambda = \lambda(d) = 2d\beta$. Let also c be a coloring of G in L colors provided by Lemma 2. We want to construct a grouping \tilde{G} of G such that for all buckets $B \in V(\tilde{G})$ and all $u \neq v \in B$,

$$c(u) = c(v) \text{ and } c(u') \neq c(v') \text{ for all } u' \in N_G(u), v' \in N_G(v) \quad (3)$$

In other words, all vertices of the same bucket are of the same color while any two neighbors of such two vertices are of different colors.

For each color $i \in [L]$, we introduce an auxiliary constraint graph F_i . The vertex set of F_i is $V(F_i) = c^{-1}(i)$ and its edge set is

$$E(F_i) = \{\{u, v\} : \exists u' \in N_G(u), v' \in N_G(v), c(u') = c(v')\}. \quad (4)$$

In our construction, each bucket of \tilde{G} will be an independent set in some F_i . Note that this will immediately imply (3). The degree of any vertex $v \in V(F_i)$ is at most

$$\deg_{F_i}(v) \leq \sum_{v' \in N_G(v)} (K_{c(v), c(v')} - 1) \leq d \left(\left\lceil \frac{\beta |c^{-1}(v)|}{L} \right\rceil - 1 \right) \leq \frac{d\beta |V(F_i)|}{L} = \frac{|V(F_i)|}{2r}.$$

This means that the greedy algorithm finds a proper coloring of each F_i in at most $\frac{|V(F_i)|}{2r} + 1$ colors, which splits each F_i in at most $\frac{|V(F_i)|}{2r} + 1$ independent sets. We create a separate bucket of \tilde{G} from each independent set of each F_i . Now we show that the four conditions from the lemma statement hold.

1. For the first property, the number of independent sets in each F_i is at most $\frac{|V(F_i)|}{2r} + 1$. Thus the number of buckets in \tilde{G} is

$$|V(\tilde{G})| \leq \sum_{i \in [L]} \left(\frac{|V(F_i)|}{2r} + 1 \right) = \sum_{i \in [L]} \left(\frac{|c^{-1}(i)|}{2r} + 1 \right) = \frac{n}{2r} + L \leq \frac{n}{r},$$

since $L = \lambda r$ and $2\lambda r^2 \leq n$.

2. For the second property, by Lemma 2, the coloring c is proper in G^2 . We can convert c to a coloring $\tilde{c}: V(\tilde{G}) \rightarrow [\lambda r]$ by assigning each bucket the color of its vertices (all of them have the same color). The resulting coloring \tilde{c} is a proper coloring of \tilde{G}^2 by (3).
3. All buckets of \tilde{G} are monochromatic with respect to c , thus, each bucket $B \in V(\tilde{G})$ is an independent set in G and the third property holds.
4. Finally, by (3), there is at most one edge in G between vertices corresponding to any pair of buckets in \tilde{G} .

Thus, the constructed grouping and its coloring satisfy all conditions of the lemma. \square

Lemma 4. *There exists a polynomial time algorithm that takes an input a graph G on n vertices of maximum degree d that needs to be 3-colored and an integer $r = o(\sqrt{n})$ and finds an equisatisfiable instance (G', H') of LIST GRAPH HOMOMORPHISM, where $|V(G')| \leq \frac{n}{r}, |V(H')| \leq \gamma(d)^r$.*

Proof. Constructing the graph G' . Let G' be the grouping of G and $c: V(G') \rightarrow [L]$ be the coloring provided by Lemma 3 where $L = \lambda(d)r$. To distinguish colorings of G and G' , we call $c(B)$, for a bucket $B \in V(G')$, a label of B . Consider a bucket $B \in V(G')$, i.e., a subset of vertices of G , and a label $i \in [L]$. From item 2 of Lemma 3 we know that c is a proper coloring of $(G')^2$. This, in particular, means that there is at most one $B' \in N_{G'}(B)$ such that $c(B') = i$. Moreover, if such B' exists then, by item 4 of Lemma 3, there exists a unique $u \in B$ and unique $u' \in B'$ such that $\{u, u'\} \in E(G)$. This allows us to define the following mapping $\phi_B: [L] \rightarrow B \cup \{0\}$: $\phi_B(i) = u$ if such B' exists and $\phi_B(i) = 0$ if B has no neighbor B' of label i .

Constructing the graph H' . We now define a redundant encoding of a 3-coloring of a bucket $B \in V(G')$. Namely, let $\mu_B: (f: B \rightarrow \{1, 2, 3\}) \rightarrow \{0, 1, 2, 3\}^L$. That is, for a 3-coloring $f: B \rightarrow \{1, 2, 3\}$ of B , μ_B is a vector v of length L . For $i \in [L]$, by $v[i]$ we denote the i -th component of v . The value of $v[i]$ is defined as follows: if $\phi_B(i) = 0$ then $v[i] = 0$, otherwise $v[i] = f(\phi_B(i))$. In other words, for a given bucket B and a 3-coloring f of its vertices, for each possible label $i \in [L]$, μ_B is the color of a vertex $u \in B$ that has a neighbor in a bucket with label i , and 0 if there is no such vertex u .

We are now ready to construct the graph H' . The set of vertices of H' is defined as follows:

$$V(H') = \{(R, l): R \in \{0, 1, 2, 3\}^L \text{ and } l \in [L]\},$$

i.e., a vertex of H' is an encoding of a 3-coloring of a bucket and a label of a bucket. A bucket $B \in V(G')$ is allowed to be mapped to $(R, l) \in V(H')$ if and only if $l = c(B)$ and there is a 3-coloring f of B such that $\mu_B(f) = R$. Informally, two vertices in $V(H')$ are joined by an edge if they define two consistent 3-colorings. Formally, $\{(R_1, l_1), (R_2, l_2)\} \in E(H')$ if and only if $R_1[l_2] \neq R_2[l_1]$. Note that $|V(G')| \leq n/r$ by Lemma 3 and $|V(H')| \leq 4^L \cdot L \leq 5^L = 5^{\lambda(d)r} = \gamma(d)^r$ for $\gamma(d) = 5^{\lambda(d)}$.

Running time of the reduction. The reduction clearly takes polynomial time.

Correctness of the reduction. It remains to show that G is 3-colorable if and only if (G', H') is a yes-instance of LIST GRAPH HOMOMORPHISM.

Assume that G is 3-colorable and take a proper 3-coloring g of G . It defines a homomorphism from G' to H' in a natural way: $B \in V(G')$ is mapped to $(\mu_B(g), l(B))$. Each list constraint is satisfied by definition. To show that each edge is mapped to an edge, consider an edge $\{B, B'\} \in E(G')$. Then, by item 4 of Lemma 3 there is a unique edge $\{u, u'\} \in E(G)$ such that $u \in B, u' \in B'$. Note that B and B' are mapped to vertices (R, l) and (R', l') such that $R[l] = g(u)$ and $R'[l'] = g(u')$. Since g is a proper 3-coloring of G , $g(u) \neq g(u')$. This, in turn, means that $\{(R, l), (R', l')\} \in E(H')$ and hence the edge $\{B, B'\}$ is mapped to this edge in H' .

For the reverse direction, consider a homomorphism $h: G' \rightarrow H'$. For each bucket $B \in V(G')$, $h(B)$ defines a proper 3-coloring of B . Together, they define a 3-coloring g of G and we need to show that g is proper. Assume, to the contrary, that there is an edge $\{u, u'\} \in E(G)$ such that $g(u) = g(u')$. By item 3 of Lemma 3, u and u' belong to different buckets $B, B' \in V(G')$. By the definition of grouping, $\{B, B'\} \in E(G')$. Since h is a homomorphism, $\{(R, l), (R', l')\} := \{h(B), h(B')\} \in E(H')$. At the same time, $R[l] = g(u) = g(u') = R'[l']$ which contradicts to the fact that $\{(R, l), (R', l')\}$ is an edge in H' . \square

4 Main Theorems

4.1 Graph Homomorphism

For the proof of the first main theorem of this paper, we need the following lemma which is proved in [7, Lemma 5].

Lemma 5. *There is a polynomial-time algorithm that from an instance (G, H) of LIST GRAPH HOMOMORPHISM where $|V(G)| = n$, $|V(H)| = h \geq 3$, constructs an instance (G', H') of GRAPH HOMOMORPHISM, where $|V(G')| \leq n + s$ and $|V(H')| \leq s$ for $s < 25h^2$, such that there is a list homomorphism from G to H if and only if there is a homomorphism from G' to H' .*

We are ready to prove our main theorem about graph homomorphisms.

Theorem 1. *Let G be an n -vertex graph G and H be an $h(n)$ -vertex graph. Unless ETH fails, for any constant $D \geq 1$ there exists a constant $c = c(D) > 0$ such that for any function $3 \leq h(n) \leq n^D$, there is no time $\mathcal{O}(h^{cn})$ algorithm deciding whether there is a homomorphism from G to H .*

Proof. The outline of the proof of the theorem is as follows. Assuming that there is a “fast” algorithm for GRAPH HOMOMORPHISM, we show that there is also a “fast” algorithm solving LIST GRAPH HOMOMORPHISM, which, in turn, implies “fast” algorithm for 3-COLORING on degree 4 graphs, contradicting ETH. In what follows, we specify what we mean by “fast”.

Let $h_0 = 25^2$. If $h(n) < h_0$ for all values of n , then an algorithm with running time $\mathcal{O}(h^{cn})$ would solve 3-COLORING in time $\mathcal{O}(h_0^{cn}) = \mathcal{O}(2^{cn \log h_0})$ (recall that $h(n) \geq 3$). Therefore, by choosing a small enough constant c such that $c \log h_0 < q$, we arrive to a contradiction with Lemma 1.

From now on we assume that $h(n) \geq h_0$ for large enough values of n . Let $c = \frac{q}{8D \log \gamma}$, where q is the constant from Lemma 1, and $\gamma := \gamma(4)$ is the constant from Lemma 4. For the sake of contradiction, let us assume that there exists an algorithm \mathcal{A} deciding whether $G \rightarrow H$ in time $\mathcal{O}(h^{cn}) = \mathcal{O}(2^{cn \log h})$, where $|V(G)| = n, |V(H)| = h := h(n)$. Now we show how to solve 3-coloring on n' -vertex graphs of degree 4 in time $2^{qn'}$, which would contradict Lemma 1.

Let $r = \frac{\log h}{4D \log \gamma}$ and $n' = \frac{nr}{2}$. Let G' be an n' -vertex graph of maximum degree four that needs to be 3-colored. Using Lemma 4 we construct an instance (G_1, H_1) of LIST GRAPH HOMOMORPHISM that is satisfiable if and only if the initial graph G' is 3-colorable, and $|V(G_1)| \leq \frac{n'}{r}, |V(H_1)| \leq \gamma^r$. By Lemma 5, this instance is equisatisfiable to an instance (G, H) of GRAPH HOMOMORPHISM where $|V(H)| < 25\gamma^{2r} \leq 25h^{\frac{1}{2D}} \leq h$ (since $D \geq 1$ and $h(n) \geq h_0$), and

$$|V(G)| < \frac{n'}{r} + 25\gamma^{2r} \leq \frac{n}{2} + 25h^{\frac{1}{2D}} \leq \frac{n}{2} + 25\sqrt{n} \leq n$$

(for sufficiently large values of n).

Now, in order to solve 3-coloring for G' , we construct an instance (G, H) with $|V(G)| \leq n$ and $|V(H)| \leq h$ of GRAPH HOMOMORPHISM and invoke the algorithm \mathcal{A} on this instance. The running time of \mathcal{A} is

$$\mathcal{O}(2^{cn \log h}) = \mathcal{O}(2^{\frac{2cn'}{r} \log h}) = \mathcal{O}(2^{2cn' \log h \cdot \frac{4D \log \gamma}{\log h}}) = \mathcal{O}(2^{8cDn' \log \gamma}) = \mathcal{O}(2^{qn'})$$

and hence we can find a 3-coloring of G' in time $\mathcal{O}(2^{qn'})$, which contradicts ETH. \square

4.2 Subgraph Isomorphism

To prove the theorem about subgraph isomorphisms, we need the following theorem of Cygan, Pachocki, and Socała (Theorem 1.3 in [5]).

Theorem 2. *Given an instance (G, H) of GRAPH HOMOMORPHISM one can in $\text{poly}(n)2^n$ time create 2^n instances of SUBGRAPH ISOMORPHISM with n vertices, where $n = |V(G)| + |V(H)|$, such that (G, H) is a yes-instance if and only if at least one of the created instances of SUBGRAPH ISOMORPHISM is a yes-instance.*

Combining Theorem 1 with Theorem 2, we immediately obtain the following theorem.

Theorem 3. *Unless ETH fails, there exists a constant $c > 0$ such that there is no algorithm deciding whether a given n -vertex graph G contains a subgraph isomorphic to a given n -vertex graph H in time $\mathcal{O}(n^{cn})$.*

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