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# Colouring quadrangulations of projective spaces

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ABSTRACT

A graph embedded in a surface with all faces of size 4 is known as a quadrangulation. We extend the definition of quadrangulation to higher dimensions, and prove that any graph $G$ which embeds as a quadrangulation in the real projective space $P^n$ has chromatic number $n + 2$ or higher, unless $G$ is bipartite. For $n = 2$ this was proved by Youngs (1996) [20]. The family of quadrangulations of projective spaces includes all complete graphs, all Mycielski graphs, and certain graphs homomorphic to Schrijver graphs. As a corollary, we obtain a new proof of the Lovász–Kneser theorem.



## 1. Introduction

A graph which embeds in the real projective plane $P^2$ so that every face is bounded by a walk of length 4 is called a (2-dimensional) *projective quadrangulation*. A remarkable

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result of Youngs [20] asserts that the chromatic number of a projective quadrangulation is either 2 or 4. On the last page of [20], Youngs notes:

> ... it would be equally worthwhile to increase the chromatic number [of the graphs in question]. A possible step in this direction is to jump from a (2-dimensional) projective plane to a higher dimensional projective space. This may not be a fruitful path to follow, and the only evidence the author can suggest in its favor is that the 5-chromatic Mycielski graphs embed pleasantly in projective 3-space in a similar fashion to their 4-chromatic counterparts in 2-space.

In this paper, we show that Youngs' intuition was correct as we extend the lower bound in his theorem to the $n$-dimensional real projective space $P^n$. To do so, we extend the notion of quadrangulation to higher dimensions as follows (for definitions, see Section 2).

Let $K$ be a generalised simplicial complex (there may be more than one simplex with the same set of vertices, unlike in the usual simplicial complex). A *quadrangulation* of $K$ is a spanning subgraph $G$ of its 1-skeleton $K^{(1)}$ such that every (inclusionwise) maximal simplex of $K$ induces a complete bipartite subgraph of $G$ with at least one edge. If the polyhedron of $K$ is homeomorphic to a topological space $X$, we say that the natural embedding of $G$ in $X$ is a quadrangulation of $X$.

Note that if $K$ triangulates the projective plane, then a quadrangulation of $K$ is a projective quadrangulation according to the usual definition recalled at the beginning of this section. Conversely, given a projective quadrangulation $H$, we can triangulate its faces and obtain $H$ as a quadrangulation of the resulting generalised simplicial complex. More precisely, this is true if none of the faces of $H$ contains a crosscap; otherwise, two edges of $H$ will be doubled in the process. However, this difference between the two definitions is unimportant as long as we are interested in vertex colouring.

Our main result is the following generalisation of the lower bound of Youngs.

**Theorem 1.1.** *If $G$ is a non-bipartite quadrangulation of the $n$-dimensional projective space $P^n$, then $\chi(G) \geq n+2$.*

We show that the family of quadrangulations of projective spaces includes all complete graphs and all (generalised) Mycielski graphs. We also prove the following result about the Schrijver graph $SG(n,k)$. (Recall that a graph $G$ is *homomorphic* to a graph $H$ if there exists a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$; note that, in this case, $\chi(G) \leq \chi(H)$.)

**Theorem 1.2.** *Let $n > 2k$ and $k \geq 1$. There exists a non-bipartite quadrangulation of $P^{n-2k}$ that is homomorphic to $SG(n,k)$.*

Since the Schrijver graph $SG(n,k)$ is a subgraph of the Kneser graph $KG(n,k)$, Theorems 1.1 and 1.2 give an alternative proof of the Lovász–Kneser theorem [9], namely $\chi(KG(n,k)) \geq n-2k+2$.

It may come as a surprise that the chromatic number of quadrangulations of $P^n$ cannot be bounded from above for any $n > 2$, as the next theorem shows.

**Theorem 1.3.** *For $n \geq 3$ and $t \geq 5$, the complete graph $K_t$ embeds in $P^n$ as a quadrangulation if $t - n$ is even.*

However, we show that sufficiently 'fine' non-bipartite quadrangulations of $P^n$ are $(n+2)$-chromatic.

The rest of the paper is organised as follows. In Section 2 we introduce the basic terminology and preliminary results. In Section 3 we prove two simple lemmas about quadrangulations which will be used later. In Section 4 we prove Theorem 1.1, and briefly discuss bounds on variants of the chromatic number. In Section 5, we prove Theorem 1.3 and provide a geometric sufficient condition for $(n+2)$-colourability. In Section 6 we show how complete graphs and Mycielski graphs embed as quadrangulations in $P^n$, and use this to prove Theorem 1.2. We conclude by presenting a conjecture and two open problems in Section 7.

## 2. Topological preliminaries

Our graph theory terminology and notation is standard, and follows Bondy and Murty [2]. All graphs considered are simple, that is, have no loops and multiple edges. The vertex and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively.

For a comprehensive account of topological methods in combinatorics and graph theory, we refer the reader to Matoušek [11] or Kozlov [8]. For an introduction to algebraic topology, see Hatcher [7] or Munkres [14].

We will deal with several different kinds of simplicial complexes. By default, our complexes are generalised simplicial complexes [8, Section 2.2] (also known as regular $\Delta$-complexes [7] or simplicial cell complexes). A topological space $K$ (a subspace of some Euclidean space $\mathbb{R}^N$) is a *generalised simplicial complex* if it can be constructed inductively using the following 'gluing process'. We start with a discrete point space $K^{(0)}$ in $\mathbb{R}^N$, and at each step $i > 0$ we inductively construct the space $K^{(i)}$ by attaching a set of $i$-dimensional simplices to $K^{(i-1)}$. We call the images of the simplices involved in the construction the *faces* or *cells* of $K$. Each simplex is attached via a gluing map $f : \partial\Delta_i \to K^{(i-1)}$ that maps the interior of each face of the boundary of the standard $i$-simplex $\Delta_i$ in $\mathbb{R}^{i+1}$ homeomorphically to the interior of a face of $K^{(i-1)}$ of the same dimension. For each $i$, the set $K^{(i)}$ is called the *$i$-skeleton* of $K$. The set of *vertices* $K^{(0)}$ is also denoted by $V(K)$. All the generalised simplicial complexes in this paper have a finite number of faces.

The *polyhedron* $\|K\|$ of a generalised simplicial complex $K$ is defined as the union of all of its cells (in $\mathbb{R}^N$). We say that $K$ *triangulates* the space $\|K\|$ or any space homeomorphic to it. All triangulations will be generalised simplicial complexes unless otherwise noted.

A generalised simplicial complex $K$ is a *geometric simplicial complex* if the embedding of each face is a linear map (a linear extension of the embedding of $K^{(0)}$).

An *abstract simplicial complex* is a non-empty hereditary set system. Given a generalised simplicial complex $K$, a natural way to assign an abstract simplicial complex to it is as follows. Let $A(K)$ be the multiset of the vertex sets of all the faces of $K$. If $A(K)$ is actually a set (that is, all the faces have distinct vertex sets), then it is an abstract simplicial complex and we say that $K$ is a *realization* of $A(K)$ (or a *geometric realization* if $K$ is a geometric simplicial complex). Furthermore, we say that $\|K\|$ is the *polyhedron* of $A(K)$. It is well known that every finite abstract simplicial complex has a geometric realization, and the polyhedra of all of its realizations are homeomorphic.

If $K$ is a generalised simplicial complex in $\mathbb{R}^N$ such that for any face $\sigma$ of $K$, its central reflection $-\sigma$ is also a face of $K$, then we say that $K$ is an *(antipodally) symmetric triangulation* of $\|K\|$. Of particular importance for us will be symmetric triangulations of the unit sphere $S^n$. Furthermore, if $K$ triangulates the unit $n$-ball $B^n$ and the subcomplex corresponding to the boundary $\partial B^n = S^{n-1}$ is a symmetric triangulation of $S^{n-1}$, then we say that $K$ is a *boundary-symmetric* triangulation of $B^n$.

The proof of Theorem 1.1 relies on a special type of abstract simplicial complex associated to a graph, which we shall now define. Given a graph $G$, the set of *common neighbours* of a set $A \subseteq V(G)$ is defined as

$$\mathrm{CN}(A) = \bigl\{v \in V(G) : \{a, v\} \in E(G) \text{ for all } a \in A\bigr\}.$$

The *box complex* of a graph $G$ without isolated vertices is the simplicial complex with vertex set $V(G) \times \{1, 2\}$, defined as

$$B(G) = \bigl\{A_1 \uplus A_2 : A_1, A_2 \subseteq V(G), A_1 \subseteq \mathrm{CN}(A_2) \neq \emptyset, A_2 \subseteq \mathrm{CN}(A_1) \neq \emptyset\bigr\},$$

where we use the notation $A \uplus B$ for the set $(A \times \{1\}) \cup (B \times \{2\})$.

Let $K$ be a generalised simplicial complex and $p$ a non-negative integer. Restricting to $\mathbb{Z}_2$ coefficients, recall that a *p-chain* of $K$ is a (finite) formal sum of some of the $p$-simplices of $K$, and the group of $p$-chains of $K$ is denoted by $C_p(K, \mathbb{Z}_2)$. The *boundary* of a $p$-chain $c$ is denoted by $\partial_p(c)$. The group of *p-cycles* is defined as $Z_p(K, \mathbb{Z}_2) = \ker \partial_p$ and the group of *p-boundaries* of $C_p(K, \mathbb{Z}_2)$ as $B_p(K, \mathbb{Z}_2) = \operatorname{Im} \partial_p$. The *p-th homology* group is the quotient $H_p(K, \mathbb{Z}_2) = Z_p(K, \mathbb{Z}_2)/B_p(K, \mathbb{Z}_2)$. Two $p$-cycles $c_1, c_2 \in Z_p(K, \mathbb{Z}_2)$ are *homologous* if they are in the same class of $H_p(K, \mathbb{Z}_2)$, i.e., if there exists a $(p+1)$-chain $d$ such that $c_1 + c_2 = \partial_{p+1}(d)$.

A homeomorphism $\nu : X \to X$ is called a *$\mathbb{Z}_2$-action* on $X$ if $\nu^2 = \nu \circ \nu = \mathrm{id}_X$. The $\mathbb{Z}_2$-action $\nu$ is *free* if it has no fixed points. A topological space $X$ equipped with a (free) $\mathbb{Z}_2$-action $\nu$ is a *(free) $\mathbb{Z}_2$-space*. A canonical example of a free $\mathbb{Z}_2$-space is $(S^n, \nu)$, where $S^n$ is the unit $n$-sphere and $\nu$ is the *antipodal action* given by $\nu : x \mapsto -x$. The box complex is equipped with a natural free $\mathbb{Z}_2$-action $\nu$ which interchanges the two copies of $V(G)$, namely $\nu : (v, 1) \mapsto (v, 2)$ and $\nu : (v, 2) \mapsto (v, 1)$. Given $\mathbb{Z}_2$-spaces $(X, \nu)$ and

$(Y, \omega)$, a continuous map $f : X \to Y$ such that $f \circ \nu = \omega \circ f$ is known as a $\mathbb{Z}_2$-*map*. If there exists a $\mathbb{Z}_2$-map from $X$ to $Y$, we write $X \xrightarrow{\mathbb{Z}_2} Y$. The $\mathbb{Z}_2$-*coindex* of $X$ is defined as

$$\operatorname{coind}(X) = \max\{n \geq 0 : S^n \xrightarrow{\mathbb{Z}_2} X\}.$$

When $K$ is a generalised simplicial complex, we write $\operatorname{coind}(K)$ instead of $\operatorname{coind}(\|K\|)$.

The main tool in the proof of Theorem 1.1 is the following inequality, which may be traced to Lovász's seminal paper [9] (where it was stated in terms of the connectivity of the neighbourhood complex). Its proof relies on the Borsuk–Ulam theorem [3]. (See also Theorem 5.9.3 in [11] and the discussion on page 99 therein.)

**Theorem 2.1.** *If $G$ is a graph without isolated vertices, then $\chi(G) \geq \operatorname{coind}(B(G)) + 2$.*

## 3. Quadrangulations

Recall the definition of a quadrangulation of $X$ from the Introduction, namely it is a subgraph $G$ of the 1-skeleton $K^{(1)}$ of a generalised simplicial complex $K$ such that $\|K\| \cong X$, and every maximal simplex of $K$ induces a complete bipartite subgraph of $G$ with at least one edge.

We define the *parity* of a cycle in a graph to be the parity of its length; a cycle is even (resp. odd) if it has even (resp. odd) parity. We start by a proving the following crucial property of quadrangulations of projective spaces.

**Lemma 3.1.** *In every quadrangulation $G$ of a topological space $X$, homologous cycles have the same parity; in particular, 0-homologous cycles are even. If $X = P^n$ and $G$ is not bipartite, then every 1-homologous cycle is odd.*

**Proof.** Let $K$ be a generalised simplicial complex whose polyhedron is homeomorphic to $X$, such that $G$ is a quadrangulation of $K$. If $a$ and $b$ are homologous 1-cycles in $G$, we may write $a + b = \partial_2(c)$, for some 2-chain $c \in C_2(K, \mathbb{Z}_2)$. By the definition of quadrangulation, every 2-simplex in $K$ is incident with an even number of edges of $G$ (namely 0 or 2), so the boundary $\partial_2(c)$ is incident with an even number of edges of $G$. Hence, the parity of the length of $a$ and $b$ is the same. It also follows that 0-homologous cycles are even.

We prove the last assertion of the lemma. It is well known (cf. [14]) that $H_1(P^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$, so there are only two $\mathbb{Z}_2$-homology classes of cycles in $P^n$: the 0- and 1-homologous cycles, which correspond to the contractible and non-contractible cycles, respectively. If $G$ is a non-bipartite quadrangulation of $P^n$, it contains at least one odd cycle, which must be 1-homologous. Hence, by the first part of the lemma, every 1-homologous cycle is odd. □

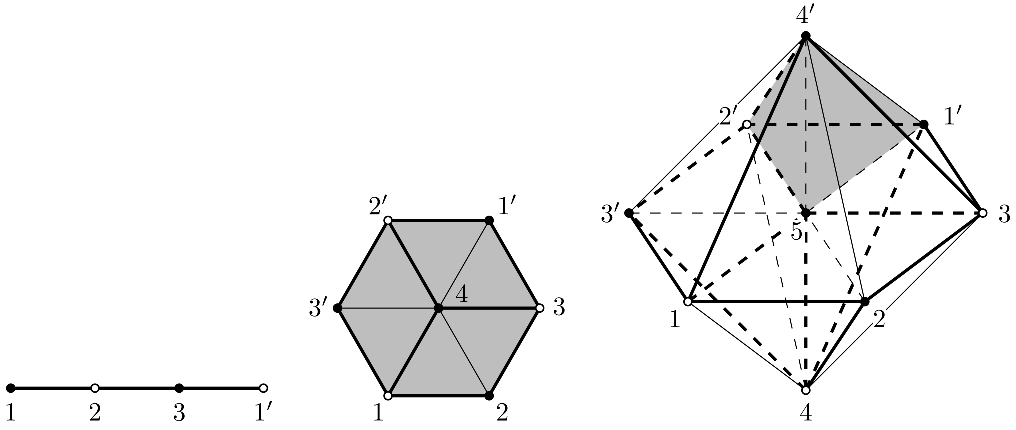


**Fig. 3.1.** Complete graphs $K_n$ ($3 \leq n \leq 5$) as quadrangulations of $P^{n-2}$, using a triangulation of $B^{n-2}$ as in Lemma 3.2(c). Thick lines are the edges of the quadrangulation, simplices are depicted in grey (only one is shown in the triangulation of $B^3$; note how the quadrangulation induces a $K_{1,3}$ on it). Dashed lines represent visibility. We write $v'$ for the antipode of the vertex $v$.

A 2-colouring $c$ of a complex $K$ is an arbitrary assignment of two colours, say black and white, to the vertices of $K$. We say that $c$ is *proper* if there is no monochromatic maximal simplex. The graph *associated* to the 2-colouring is a spanning subgraph of the 1-skeleton of $K$ consisting of all edges with one end white and the other black.

The following lemma (which will also be needed later on) makes it easier to draw examples of quadrangulations using triangulations of $B^n$ symmetric on the boundary. Fig. 3.1 shows, using condition (c), how the complete graphs $K_3$, $K_4$ and $K_5$ embed as quadrangulations in $P^1$, $P^2$ and $P^3$, respectively.

**Lemma 3.2.** *For a graph $G$, consider the following statements:*

(a) *$G$ is a non-bipartite quadrangulation of $P^n$.*
(b) *There is a symmetric triangulation $T$ of $S^n$ such that no simplex of $T$ contains antipodal vertices, and there is a proper antisymmetric 2-colouring of $T$ such that $G$ is obtained from the associated graph by identifying all pairs of antipodal vertices.*
(c) *There is a boundary-symmetric triangulation $T'$ of $B^n$ such that no simplex of $T'$ contains antipodal boundary vertices, and there is a proper boundary-antisymmetric 2-colouring of $T'$ such that $G$ is obtained from the associated graph by identifying all pairs of antipodal boundary vertices.*

*Statements (a) and (b) are equivalent and are implied by statement (c).*

**Proof.** We start with the implication (a) $\Longrightarrow$ (b). If $G$ is a quadrangulation of $P^n$, then by definition there exists a generalised simplicial complex $K$ such that $\|K\|$ is homeomorphic to $P^n$, $G$ is a subgraph of $K^{(1)}$ and every maximal simplex of $K$ induces a complete bipartite subgraph of $G$ with at least one edge. It is well known (cf. [14]) that

$P^n$ is the quotient space $S^n/\nu$ where $\nu$ is the antipodal action on $S^n$, and the projection of $\nu$ is a covering map $p : S^n \to S^n/\nu$. Therefore, there is a corresponding simplicial covering map $q : T \to K$, where $(T, \xi)$ is a free generalised simplicial $\mathbb{Z}_2$-complex such that $\|T\| \cong S^n$, and the homeomorphism induces a centrally symmetric generalised simplicial complex structure on $S^n$, with the $\mathbb{Z}_2$-action given by the antipodal map. The graph $\tilde{G} = q^{-1}(G)$ is easily seen to be a quadrangulation of $T$. Since all cycles in $\tilde{G}$ are 0-homologous in the $n$-sphere $\|T\|$, $\tilde{G}$ is bipartite by Lemma 3.1. Fix a 2-colouring $c : V(\tilde{G}) \to \{1, 2\}$; this defines a colouring of $T$ such that $\tilde{G}$ is its associated graph. By Lemma 3.1, $c$ is antisymmetric. Suppose that $c$ is not proper. Then there exists an $n$-simplex $\sigma \in T$ such that $c(u) = 1$ for every $u \in \sigma$ (and by antisymmetry, $c(-u) = 2$ for all such $u$). Therefore, the vertices of $\sigma$ and $-\sigma$ form independent sets of $\tilde{G}$. Consequently, the vertices of the $n$-simplex $q(\sigma) \in K$ form an independent set of $G$, contradicting the assumption that $G$ is a quadrangulation of $P^n$. Hence $c$ is proper, as required. Finally, the fact that no simplex of $T$ contains antipodal vertices follows immediately from the fact that $K = T/\nu$ has no loops.

We now prove (b) $\Longrightarrow$ (a). Let $T$ be a triangulation and $c$ a 2-colouring of $T$ as in property (b), and let $G_c$ be the associated graph of $c$. The quotient of the $\mathbb{Z}_2$-action on $T$ is a triangulation $K$ of $P^n$. The graph $G$, being obtained from $G_c$ by identifying antipodal vertices, is a subgraph of the 1-skeleton of $K$.

We claim that the subgraph $H$ of $G$ induced on the vertices of any maximal simplex $\sigma$ of $K$ is a complete bipartite graph with at least one edge. Let $\tau$ be a simplex of $T$ mapped to $\sigma$ by the covering map corresponding to the $\mathbb{Z}_2$-action on $T$. Then $H$ is isomorphic to the subgraph of $G_c$ induced on the vertices of $\tau$; by the definition of the associated graph, $H$ is complete bipartite. Moreover, since $c$ is proper, $H$ has at least one edge.

Finally, for the implication (c) $\Longrightarrow$ (b), we take two copies of $T'$ (say $T'_1$ and $T'_2$), retain the given 2-colouring on $T'_1$, and invert it on $T'_2$. We glue $T'_1$ and $T'_2$ together by identifying each simplex of the boundary of $T'_1$ with the antipode of its copy in $T'_2$. The resulting generalised simplicial complex is a symmetric triangulation $T'_{12}$ of $S^n$. Furthermore, note that if the simplices being identified are vertices, then they have the same colour. It follows that the original proper 2-colouring $c'$ of $T'$ induces a proper 2-colouring $c'_{12}$ of $T'_{12}$. If we identify the antipodal vertices in the associated graph of $c'_{12}$, we obtain the same graph (namely $G$) as if we identify the antipodal boundary vertices in the associated graph of $c'$. Lastly, note that no simplex of $T'_{12}$ contains antipodal vertices: otherwise, by the construction, there would be a simplex of $T'_1$ containing antipodal boundary vertices, contrary to the assumption. □

It is natural to ask whether statement (c) of Lemma 3.2 is equivalent to statements (a) and (b). This question seems to be open (cf. the discussion in the last paragraph of [16]).

## 4. A lower bound on the chromatic number

This section is devoted to the proof of the first of our results mentioned in Section 1:

**Theorem 1.1.** *If $G$ is a non-bipartite quadrangulation of $P^n$, then $\chi(G) \geq n+2$.*

**Proof.** By Lemma 3.2, there are a symmetric triangulation $T$ of $S^n$ and an antisymmetric 2-colouring $c : V(T) \to \{1,2\}$ such that $G$ is obtained from the associated graph by identifying antipodal pairs of vertices. Define the mapping $f : V(T) \to V(B(G))$ as $f : v \mapsto (v, c(v))$.

Let $A$ be the set of vertices of an arbitrary simplex in $T$. Set $A_i = A \cap c^{-1}(i)$; so $f(A) = A_1 \uplus A_2$. To prove that $f$ is a simplicial map, it suffices to show that $A_1 \uplus A_2$ is a simplex of $B(G)$. Let $A' \in T$ be the vertex set of a maximal simplex in $T$ such that $A \subseteq A'$, and define $A'_i = A' \cap c^{-1}(i)$ for $i = 1,2$. By the definition of quadrangulation, $A_i \subseteq A'_i \subseteq \mathrm{CN}(A'_{3-i}) \subseteq \mathrm{CN}(A_{3-i})$ and $A'_i \neq \emptyset$, where $i = 1,2$. This shows that $A_1 \uplus A_2 \in B(G)$, so $f$ is indeed a simplicial map.

Moreover, if $\xi$ and $\omega$ are the $\mathbb{Z}_2$-actions on $T$ and $B(G)$, respectively, and $v$ is a vertex in $T$, then $f(\xi(v)) = (v, 3-c(v)) = \omega(v, c(v)) = \omega(f(v))$, so $f \circ \xi = \omega \circ f$. This shows that $f$ is a simplicial $\mathbb{Z}_2$-map, which extends naturally to a simplicial $\mathbb{Z}_2$-map $f' : \mathrm{sd}(T) \to \mathrm{sd}(B(G))$; note that $\mathrm{sd}(T)$ is a simplicial complex. Its affine extension is therefore a continuous $\mathbb{Z}_2$-map $\|f'\| : S^n \to \|B(G)\|$, so $\mathrm{coind}(B(G)) \geq \mathrm{coind}(S^n) = n$. Since $G$ clearly has no isolated vertices, the result follows by applying Theorem 2.1. □

Notice that in the proof of Theorem 1.1, we have in fact shown that $\mathrm{coind}(B(G)) \geq n$ for any non-bipartite quadrangulation $G$ of $P^n$. In conjunction with results of Simonyi and Tardos [18,19], this implies that any non-bipartite quadrangulation $G$ of $P^n$ satisfies the following properties (for definitions, see [18]):

1. $G$ has local chromatic number at least $\lceil n/2 \rceil + 2$;
2. when $n$ is even, $G$ has circular chromatic number at least $n+2$;
3. in any proper colouring of $G$, there is a copy of $K_{\lfloor \frac{n+2}{2} \rfloor, \lceil \frac{n+2}{2} \rceil}$ in which all vertices receive different colours;
4. in any proper $(n+2)$-colouring of $G$, there is a copy of $K_{\ell,m}$ in which all vertices receive different colours, for any $\ell, m \geq 1$ such that $\ell + m = n+2$.

For $n = 2$ these facts were shown in [13,4,1,19], respectively.

## 5. Upper bounds on the chromatic number

Every non-bipartite quadrangulation of the projective plane is 4-chromatic; that is, the lower bound proved by Youngs [20] (or given by Theorem 1.1) is actually the right value. This is not the case in higher dimensions; as the following result shows, the chromatic number of quadrangulations of $P^3$ is unbounded.

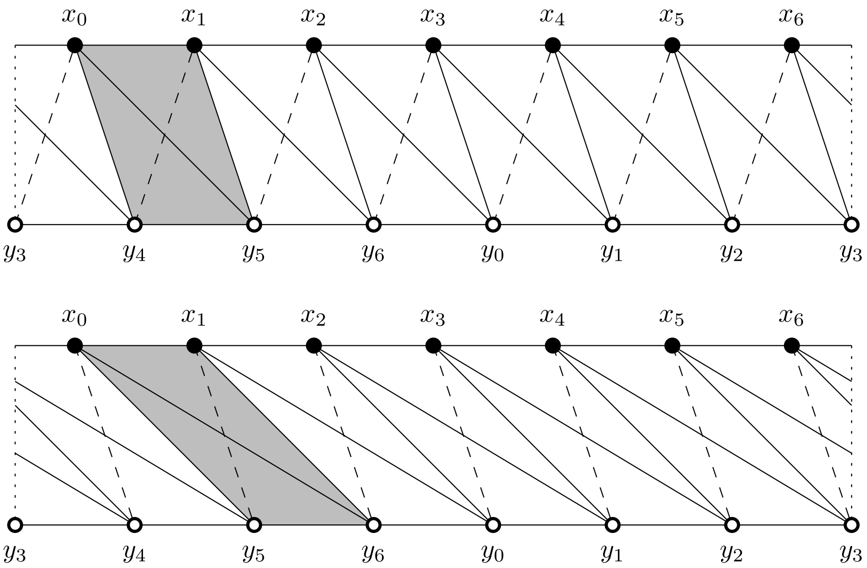


**Fig. 5.1.** The complexes $L_1$ and $L_2$ in the proof of Theorem 5.1 for $r = 3$. The defining 3-simplices $[x_0, x_1, y_4, y_5]$ and $[x_0, x_1, y_5, y_6]$ are shown grey. Dashed lines represent visibility from the origin. The vertical dotted lines are identified.

**Theorem 5.1.** *For all $r \geq 3$, the complete graph $K_{2r+3}$ embeds in $P^3$ as a quadrangulation.*

**Proof.** Consider the cylinder $C$ in $\mathbb{R}^3$ that is the product of the unit circle $S^1$ in the $xy$ plane with the interval $[-1, 1]$ on the $z$ axis. Let $C^+$ denote the top circle of $C$ and $C^-$ its bottom circle. Distribute points $x_0, \dots, x_{2r}$ evenly along $C^+$ and colour them black. Further, distribute white points $y_0, \dots, y_{2r}$ along $C^-$ in such a way that for each $i$, the vertical projection of $x_i$ to $C^-$ is antipodal to $y_i$ on $C^-$. Let $V = \{x_0, \dots, x_{2r}, y_0, \dots, y_{2r}\}$. For any geometric simplex $\tau$ on $V$, define $\Theta(\tau)$ as the geometric simplicial complex on $V$ whose facets are all the images of $\tau$ under rotations of $\mathbb{R}^3$ about the $z$ axis mapping $V$ to itself.

For $j = 1, \dots, r-1$, let $\sigma_j$ be the linear simplex $[x_0, x_1, y_{r+j}, y_{r+j+1}]$, and let $L_j = \Theta(\sigma_j)$. (See Fig. 5.1 for an illustration.) Further, define $Z^+ = \Theta([x_0, x_1])$ and $Z^- = \Theta([y_0, y_1])$.

Let us define the *inner boundary* of any geometric simplicial complex in $\mathbb{R}^3$ as the subcomplex consisting of faces fully visible from the origin. We claim that the inner boundary of $L_j$ is the complex $L_j^* = \Theta([x_0, x_1, y_{r+j+1}]) \cup \Theta([x_0, y_{r+j}, y_{r+j+1}])$. (To see this, project $\sigma_j$ to the $xy$ plane and note that among the edges of $\sigma_j$ with one end white and one end black, $[x_0, y_{r+j+1}]$ is the one closest to the $z$ axis; cf. Fig. 5.2. Consequently, the inner boundary consists of the 2-simplices containing it, together with their rotational images.)

Let $j < \ell$ be distinct integers between 1 and $r-1$. It is easy to check that

$$L_j \cap L_\ell = \begin{cases} L_j^* & \text{if } \ell = j+1, \\ Z^+ \cup Z^- & \text{otherwise.} \end{cases}$$

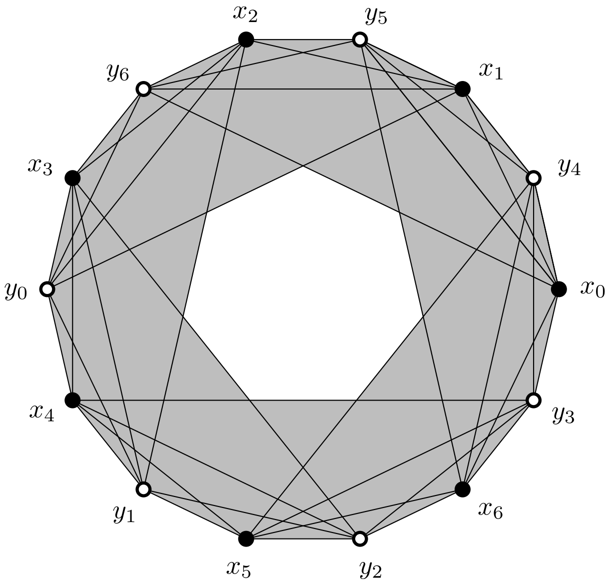


**Fig. 5.2.** The vertical projection of $L_1 \cup L_2$ (grey) and its 1-skeleton in the proof of Theorem 5.1 (for $r = 3$).

Let $L$ be the union of all the complexes $L_j$. By the above, $L$ is homotopy equivalent to $C$ and its inner boundary is $L^*_{r-1}$.

To make $L$ into a triangulation of the 3-dimensional ball, we do the following:

- place a white vertex $x$ in the centre of $C^+$ and a black vertex $y$ in the centre of $C^-$,
- add the joins $x * Z^+$ and $y * Z^-$,
- place a black vertex at the origin and add its join with $L^*_{r-1} \cup (x * Z^+) \cup (y * Z^-)$.

The resulting complex triangulates the convex hull of $C$, which is homeomorphic to $B^3$. It corresponds to a triangulation of $B^3$ which is symmetric on the boundary; furthermore, the 2-colouring defined above is proper and antisymmetric on the boundary. The associated graph, with antipodal boundary vertices identified, is easily seen to be the complete graph $K_{2r+3}$. □

We obtain the following result as a corollary of Theorem 5.1 for higher dimensions:

**Theorem 1.3.** *For $n \geq 3$ and $t \geq 5$, the complete graph $K_t$ embeds in $P^n$ as a quadrangulation if $t - n$ is even.*

**Proof.** Using Theorem 5.1, embed $K_{t-n+3}$ as a quadrangulation in $P^3$. Let $T$ be a 2-coloured triangulation of $S^3$ satisfying the condition of Lemma 3.2(b). Taking the $(n-3)$-fold suspension, that is, the $(n-3)$-fold join of $T$ with $S^0$ (consisting of two points, one black and one white), we obtain a triangulation $T'$ of $S^n$, again satisfying the condition of Lemma 3.2(b). Moreover, identifying the antipodal vertices in the associated graph, we obtain $K_t$. $\square$

While unbounded in general in dimension $n > 2$, the chromatic number of higher-dimensional projective quadrangulations is bounded when the quadrangulation is 'sufficiently fine', as shown by the following proposition.

**Proposition 5.2.** *Let $G$ be a quadrangulation of $P^n$ and $T$ the corresponding 2-coloured symmetric triangulation of $S^n$. If the vertices of $T$ lie on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ and adjacent vertices of different colours are at Euclidean distance less than $\frac{2}{\sqrt{n+3}}$, then $\chi(G) = n + 2$.*

Before proving the proposition, let us recall the following construction due to Erdős and Hajnal [5]. The *Borsuk graph* $B(n, \alpha)$ is defined as the (infinite) graph whose vertices are the points of $\mathbb{R}^n$ on $S^{n-1}$, and the edges connect points at Euclidean distance at least $\alpha$, where $0 < \alpha < 2$. By the Borsuk–Ulam theorem [3], $\chi(G) \geq n + 1$ (in fact the two statements are equivalent, as noted by Lovász [10]). By using the standard $(n + 1)$-colouring of $S^{n-1}$ (based on the central projection of a regular $n$-simplex), it may be shown that $B(n, \alpha)$ is $(n + 1)$-chromatic for all $\alpha$ sufficiently large. In particular, Simonyi and Tardos [18] have shown that if $\alpha > 2\sqrt{1 - 1/(n + 2)}$ then $B(n, \alpha_0)$ is $(n + 1)$-chromatic.

**Proof of Proposition 5.2.** Consider the symmetric triangulation $T$ and identify the vertices of $G$ with their corresponding black vertices in $\|T\|$. This gives an embedding of $G$ in $\mathbb{R}^{n+1}$ with all vertices on $S^n$, such that adjacent vertices are 'nearly antipodal'. More precisely, the Euclidean distance between them is greater than $\alpha_0$, where

$$\alpha_0 = \sqrt{4 - \left(\frac{2}{\sqrt{n+3}}\right)^2} = 2\sqrt{1 - \frac{1}{n+3}}.$$

In particular, $G$ is a subgraph of the Borsuk graph $B(n+1, \alpha)$, where $\alpha > \alpha_0$. As shown by Simonyi and Tardos [18], $\chi(B(n + 1, \alpha)) \leq n + 2$, so a fortiori $\chi(G) \leq n + 2$. $\square$

## 6. Application to Kneser and Mycielski graphs

Recall that for given $n, k \geq 1$, where $n > 2k$, the vertices of the *Kneser graph* $KG(n, k)$ are the $k$-element subsets of $[n] = \{1, \ldots, n\}$, and edges join pairs of subsets that are disjoint.

Schrijver [17] characterised a family of vertex-critical subgraphs of Kneser graphs. Viewing $[n]$ as the vertex set of the $n$-cycle $C_n$ (with edges $12, 23, \ldots, (n - 1)n, n1$), let $\mathcal{I}_k(C_n)$ be the collection of all $k$-element independent subsets of $C_n$. The *Schrijver graph* $SG(n, k)$ is defined as the induced subgraph of $KG(n, k)$ on $\mathcal{I}_k(C_n)$. As shown in [17], $SG(n, k)$ is $(n - 2k + 2)$-chromatic and vertex-critical.

In this section, we prove the following result, stated in Section 1:

**Theorem 1.2.** *Let $n > 2k$ and $k \geq 1$. There exists a non-bipartite quadrangulation of $P^{n-2k}$ that is homomorphic to $SG(n,k)$.*

In view of Theorem 1.1, Theorem 1.2 gives an alternative proof that $\chi(SG(n,k)) \geq n - 2k + 2$.

To obtain the quadrangulation in Theorem 1.2, we utilise the generalised Mycielski construction of Gyárfás, Jensen and Stiebitz [6] (see also [11, p. 133]). Let $r \geq 1$ and let $G$ be a graph with vertex set $V$. The *generalised Mycielskian* $M_r(G)$ of $G$ is defined as follows:

(M1) the vertex set of $M_r(G)$ is $\{z\} \cup (V \times [r])$,
(M2) for $i = 2, \dots, r$ and adjacent vertices $v, w$ of $G$, the vertices $(v, i)$ and $(w, i-1)$ of $M_r(G)$ are adjacent,
(M3) $M_r(G)$ contains a copy of $G$ on $V \times \{r\}$,
(M4) the vertex $z$ (which we will refer to as the *universal* vertex) is adjacent to all vertices in $V \times \{1\}$.

(We note that the original definition was given for $r \geq 2$; we extend it to the case $r = 1$, which consists in adding the universal vertex.)

It is not difficult to see that for all $r \geq 1$, the chromatic number of $M_r(G)$ is at most $\chi(G) + 1$. While there are graphs for which the inequality is strict, it follows from the results of [6] that the equality holds for graphs obtained from an odd cycle by a finite number of iterations of the generalised Mycielskian $M_r(\cdot)$.

For $n \geq 3$, the *Mycielski graph* $M_n$ is defined as the graph obtained from the 5-cycle by the $(n-3)$-fold iteration of the operation $M_2(\cdot)$. In particular, $M_3$ is the 5-cycle. It is well known [15] that for each $n$, $M_n$ is triangle-free and $n$-chromatic. The following theorem gives a new geometrical intuition for this fact:

**Theorem 6.1.** *If $G$ is a nonbipartite quadrangulation of $P^n$ and $r \geq 1$, then $M_r(G)$ is a nonbipartite quadrangulation of $P^{n+1}$. In particular, the Mycielski graph $M_n$ and the complete graph $K_n$ embed as nonbipartite quadrangulations in $P^{n-2}$.*

**Proof.** By Lemma 3.2, let $T$ be an antipodally symmetric triangulation of $S^n$ with an antisymmetric 2-colouring such that $G$ is obtained from the associated graph by identifying antipodal vertices.

We will extend $T$ to a triangulation $L$ of the ball $B^{n+1}$ (with a proper 2-colouring) whose associated graph, with antipodal boundary vertices identified, is $M_r(G)$. The construction can be viewed as a counterpart of the Mycielski construction on the level of simplicial complexes.

To avoid excessive formalism, we will assume that $T$ is a geometric simplicial complex with a realisation $\|T\|$ in $\mathbb{R}^{n+1}$ where all the vertices lie on the unit $n$-sphere $S^n$. The

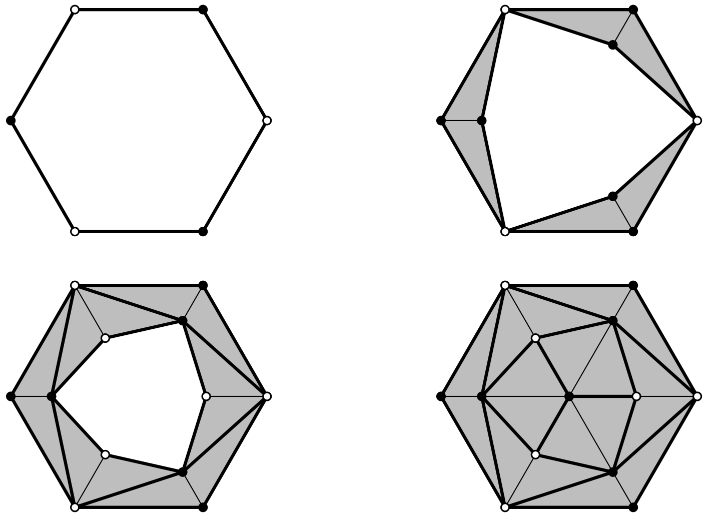

**Fig. 6.1.** The complex $T = L_3$ and the complexes $L_2$, $L_1$ and $L_0 = L$ in the proof of Theorem 6.1 (for $n = 1$ and $r = 3$). The thick edges are the edges of the associated graph, the grey regions represent 2-simplices.

general case (where we have a generalised simplicial complex and the simplices are not necessarily linear) follows using the same idea.

Let us fix an arbitrary linear order $\preceq$ on the vertices of $G$ (the *order of precedence*). We construct a sequence $L_r, \dots, L_1, L_0$ of geometric complexes with proper 2-colourings starting with $L_r = T$, ending with $L_0 = L$, and such that for each $i = r, \dots, 1$, $L_i$ is a subcomplex of $L_{i-1}$ (as a 2-coloured complex).

The complex $L_r$ has two antipodal vertices of different colours for each vertex $v \in V(G)$; let us call the white one $v^r$ and the black one $v^{r+1}$. For $1 \leq i \leq r-1$, the set $V(L_i) - V(L_{i+1})$ is denoted by $V^i$ and consists of vertices $v^i$, where $v \in V(G)$. We define $V^r = V(L_r)$ and $V^0 = \{z\}$, where $z$ is a special vertex. For each vertex of $V^i$ ($0 \leq i \leq r$), the integer $i$ is its *level*.

For each $i = r-1, \dots, 1$, we extend $L_{i+1}$ to $L_i$ as follows. (See the illustration in Figs. 6.1 and 6.2.) Define the set of *active* vertices as $V^{i+2} \cup V^{i+1}$. For each vertex $v$ of $G$ in the order of precedence, do the following:

- add the vertex $v^i$, embed it in the open segment from $v^{i+2}$ to the origin and assign to it the colour of $v^{i+2}$,
- for each simplex $\sigma$ containing $v^{i+2}$ and consisting of active vertices, add the simplex $\sigma \cup \{v^i\}$ (with the linear embedding),
- mark $v^i$ active and $v^{i+2}$ inactive.

Note that the colour of each vertex of $L$ only depends on its level, and it alternates as the level changes from $r$ to 0.

As a final step, construct the complex $L_0 = L$:

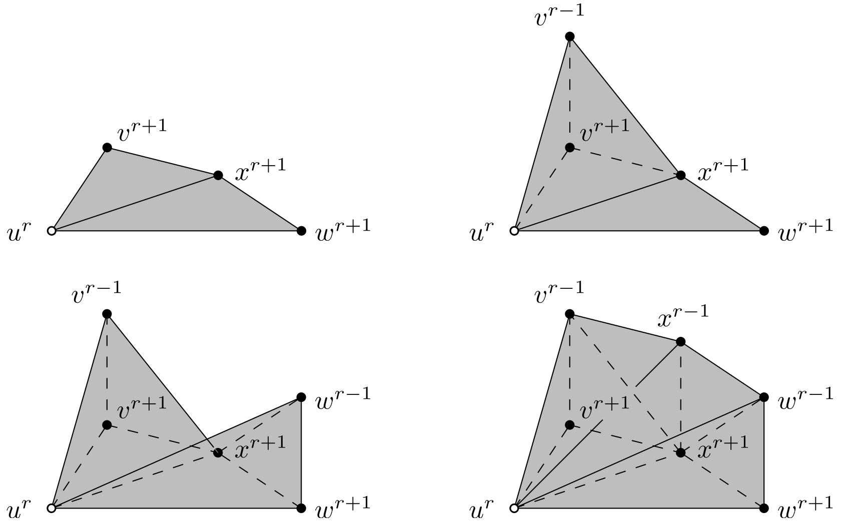


**Fig. 6.2.** Adding 3-simplices in the construction of $L_{r-1}$ in the proof of Theorem 6.1 (for $n = 2$, assuming $v \preceq w \preceq x$, where $v, w, x \in V(G)$). Dashed lines indicate visibility, the grey regions represent simplices of dimension 2 and 3.

- add the vertex $z$, placing it at the origin, and assign to it the colour given to the vertices in $V^2$,
- for each simplex $\sigma$ of $L_1[V_2 \cup V_1]$, add the simplex $\sigma \cup \{z\}$ to $L_0$.

Clearly, $L$ is an $(n + 1)$-dimensional complex containing $T$ as a subcomplex. We claim that $L$ triangulates the $(n+1)$-dimensional ball $B^{n+1}$. Indeed, each complex $L_i$ in the sequence is obtained from the preceding one by 'thickening' the simplices of $L_{i+1}[V^{i+2} \cup V^{i+1}]$ at the vertices of $V^{i+2}$, in the direction to the origin. After a series of these steps, the thickened sphere $L_1$ is filled in by inserting the vertex $z$ and joining it to all the simplices forming the 'interior boundary' of $L_1$.

Let $\tilde{G}$ be the graph obtained from the associated graph of $L$ by identifying antipodal boundary vertices. We show that $\tilde{G} = M_r(G)$. To this end, we check conditions (M1)–(M4) in the definition of $M_r(G)$.

The vertex set of $\tilde{G}$ is $\{z\} \cup V(G) \times [r]$, since the identification of antipodal vertices of $T$ identifies each vertex at level $r+1$ with a vertex at level $r$. (If the identification involves vertices, say, $v^{r+1}$ and $v^r$, we continue to call the resulting vertex $v^r$.) Furthermore, the induced subgraph of $\tilde{G}$ on the set of vertices arising from the identification is $G$ by the assumption on $T$. So far, we have checked conditions (M1) and (M3).

To prove (M2) and (M4), recall that the colours of vertices of $L$ alternate with respect to their level. Since the levels of vertices in each simplex of $L$ differ by at most two, all the edges of $\tilde{G}$ come from 1-simplices of $L$ of the form $v^i w^{i-1}$, where $v, w \in V(G)$ and $1 \leq i \leq r + 1$. It is easy to prove by induction that for each $i$, $2 \leq i \leq r$, and all $u, v \in V(G)$, $L$ contains the 1-simplex $[v^i, w^{i-1}]$ if and only if $v$ and $w$ are neighbours in $G$. This implies property (M2). Property (M4) follows once we note that $L$ contains

1-simplices $[z, v^1]$ for all $v \in V(G)$, they are not monochromatic, and $z$ does not form a 1-simplex with any vertex at level greater than 2.

The proof that $\tilde{G} = M_r(G)$ is complete. The statement about the Mycielski graphs and the complete graphs follows from the above by induction (and the observation that the 5-cycle and $K_3$ embed in $P^1$ as quadrangulations). $\square$

Theorem 1.2 will be derived from the following lemma:

**Lemma 6.2.** *If $k \geq 1$ and $n > 2k+1$, then the graph $M_k(SG(n-1,k))$ is homomorphic to $SG(n,k)$.*

**Proof.** We will explicitly describe a homomorphism $f$ from $M_k(SG(n-1,k))$ to $SG(n,k)$. Let $I$ be a vertex of $SG(n-1,k)$ and let $(I,1),\ldots,(I,k)$ be its copies in $M_k(SG(n-1,k))$. Furthermore, let $Z$ be the universal vertex of $M_k(SG(n-1,k))$.

Suppose that $I = \{a_1,\ldots,a_k\}$, where $a_1 < \cdots < a_k$. We first define $f(I, k-2i)$, where $0 \leq i < k/2$. The image is obtained from $I$ by replacing the first $i$ elements by the $i$ least odd numbers, and replacing the last $i$ elements by the arithmetic progression of length $i$ and step 2 ending with $n-1$. In symbols,

$$\begin{aligned} f(I, k-2i) = \{&1, 3, \ldots, 2i-1, \\ &a_{i+1}, a_{i+2}, \ldots, a_{k-i}, \\ &n-2i+1, n-2i+3, \ldots, n-1\}. \end{aligned}$$

In particular, $(I,k)$ is mapped to $I$. Since, clearly, $a_{i+1} \geq 2i+1$ and $a_{k-i} \leq n-2i-1$, we find that $f(I, k-2i)$ is a vertex of $SG(n,k)$.

Next, we define $f(I, k-2i-1)$, where $0 \leq i \leq k/2-1$. In the case that $a_1 > 1$, we set

$$\begin{aligned} f(I, k-2i-1) = \{&2, 4, \ldots, 2i, \\ &a_{i+1}, a_{i+2}, \ldots, a_{k-i-1}, \\ &n-2i, n-2i+2, \ldots, n\}. \end{aligned}$$

The definition for the case $a_1 = 1$ is almost the same, except that the element $a_{i+1}$ is replaced by $a_{k-i}$.

To complete the definition, it remains to set

$$f(Z) = \begin{cases} \{1, 3, \ldots, k-1, n-k+1, n-k+3, \ldots, n-1\} & \text{if } k \text{ is even,} \\ \{2, 4, \ldots, k-1, n-k+1, n-k+3, \ldots, n-2, n\} & \text{if } k \text{ is odd.} \end{cases}$$

Note that the image of $f$ is contained in the vertex set of $SG(n,k)$. To verify that $f$ is a homomorphism, we consider a pair of adjacent vertices of $M_k(SG(n-1,k))$ and show that their images are disjoint (that is, adjacent in $SG(n,k)$).

First, consider vertices $(I, k-2i)$ and $(J, k-2i-1)$, where $I = \{a_1, \dots, a_k\}$ and $J = \{b_1, \dots, b_k\}$ are disjoint and $0 \le i < k/2 - 1$. Suppose that $x \in f(I, k-2i) \cap f(J, k-2i-1)$. It follows that $x \ge 2i+1$ (as all the smaller elements of $f(I, k-2i)$ are odd and all the smaller elements of $f(J, k-2i-1)$ are even). Similarly, $x \le n - 2i - 1$. Then, however, the definition implies that $x \in I \cap J$, a contradiction.

A similar argument works for the vertices $(I, k-2i-1)$ and $(J, k-2i-2)$ $(0 \le i \le k/2 - 2)$. Thus, the only remaining case is the pair $Z$ and $(I, 1)$. Suppose that $k$ is even. By the definition,

$$f(I, 1) = \{2, 4, \dots, k-2, a_{k/2}, n-k+2, n-k+4, \dots, n\},$$
$$f(Z) = \{1, 3, \dots, k-1, n-k+1, n-k+3, \dots, n-1\}.$$

Since $a_{k/2} \notin \{k-1, n-k+1\}$, we find $f(I, 1) \cap f(Z) = \emptyset$. An analogous argument for odd $k$ completes the proof that $f$ is a homomorphism. $\square$

**Proof of Theorem 1.2.** By Lemma 6.2, $M_k(SG(n-1, k))$ is homomorphic to $SG(n, k)$. This implies that $M_k(M_k(SG(n-2, k)))$ is homomorphic to $SG(n, k)$ since, in general, any graph homomorphism from $H$ to $H'$ determines a homomorphism from $M_k(H)$ to $M_k(H')$.

Continuing, we find that the graph

$$M_k\big(M_k\big(\dots M_k\big(SG(2k+1, k)\big) \dots\big)\big), \tag{6.1}$$

where $M_k(\cdot)$ is applied $n-2k-1$ times, is homomorphic to $SG(n, k)$. Since $SG(2k+1, k)$ quadrangulates $P^1$, Theorem 6.1 implies that (6.1) is a non-bipartite quadrangulation of $P^{n-2k}$. $\square$

## 7. Conclusion

We conclude the paper with some open questions. In relation to Schrijver graphs, it appears likely that Theorem 1.2 can be strengthened as follows:

**Conjecture 7.1.** *For $n > 2k \ge 2$, the Schrijver graph $SG(n, k)$ contains a non-bipartite quadrangulation of $P^{n-2k}$ as a spanning subgraph.*

The result of Youngs concerning quadrangulations of $P^2$ was extended to arbitrary non-orientable surfaces by Archdeacon et al. [1] and by Mohar and Seymour [12] using the notion of odd quadrangulation.

**Problem 7.2.** *Is a similar generalisation possible in the higher-dimensional case?*

An interesting question posed by a referee is whether the geometric condition in Proposition 5.2 could be replaced by a condition on the odd girth of the quadrangulation (the length of the shortest odd cycle). In particular:

**Problem 7.3.** *Is there a constant $g$ such that every quadrangulation of $P^n$ of odd girth at least $g$ is $(n+2)$-chromatic?*

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