No-three-in-line problem on a torus: periodicity

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Basic definitions

- Discrete torus $T_{m \times n}$ of size $m \times n$ is a set $\{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \subset \mathbb{Z}^2$.
- Line on $T_{m \times n}$ is an image of a line in $\mathbb{Z}^2$ under a mapping which maps a point $(x, y) \in \mathbb{Z}^2$ to the point $(x \mod m, y \mod n)$.
- Line in $\mathbb{Z}^2 \{ (b_1, b_2) + k(v_1, v_2); k \in \mathbb{Z} \}$, where $\gcd(v_1, v_2) = 1$.

Notions

- $\tau_{m,n}$ denotes the maximal number of points which can be placed on the discrete torus of sizes $m \times n$ so that no three of these points are collinear.
- $\sigma_z$ is a sequence which we obtain by fixing one of the coordinates of a torus. In other words $\sigma_z(x) := \tau_{z,x}$.
- $\pi_{m,n}$ denotes the mapping which maps a point $(x, y) \in \mathbb{Z}^2$ to the point $(x \mod m, y \mod n)$.

Main results

**Theorem 1.** The sequence $\sigma_z$ is periodic for all positive integers $z$ greater than 1.

When $z$ is a power of a prime we can say more:

**Theorem 2.** Let $T_{p^a \times p^{(a-1)p+2}}$ be a torus where $p$ is a prime and $a \in \mathbb{N}$. Then $\tau_{p^a, p^{(a-1)p+2}} = 2p^a$.

**Theorem 3.** Let $p$ be a prime, $a \in \mathbb{N}$. Let us denote $m := \min\{x; \sigma_p(x) = 2p^a\}$. Then $m = p^b$ for some $b \geq a$ and the sequence $\sigma_{p^a}$ is periodic with the period $m$.

Tools

**Theorem 4** (Chinese Remainder Theorem). Let $m, n$ be positive integers. Then two simultaneous congruences

$$x \equiv a \pmod{m},$$

$$x \equiv b \pmod{n}$$

are solvable if and only if $a \equiv b \pmod{\gcd(m, n)}$. Moreover, the solution is unique modulo $\text{lcm}(m, n)$, where $\text{lcm}$ denotes the least common multiple.

**Theorem 5** (Dirichlet’s Theorem). Let $a, b$ be positive relatively prime integers. Then there are infinitely many primes of the form $a + nb$, where $n$ is a non-negative integer.
**Theorem 6** (Langrange’s Theorem). Let $G$ be a group and $H$ its subgroup. Then $|G| = [G : H] \cdot |H|$. 

**Theorem 7.** Let $m, n \in \mathbb{N}$. Then $\tau_{m,n} \leq 2 \gcd(m, n)$. 

**Lemma 8.** Let $m, n, x, y$ be positive integers. Then $\tau_{m,n} \leq \tau_{xm,yn}$. 

**Lemma 9.** Let $m, n, x, y$ be positive integers such that $m, n$ are not both 1 and $\gcd(x, y) = \gcd(m, y) = \gcd(n, x) = 1$. Then $\tau_{m,n} = \tau_{xm,yn}$. 

**Lemma 10.** Let $z \in \mathbb{N}$ and $z = \prod_{i \in I} p_i^{a_i}$ be its prime factorization. There exists $m_z = \prod_{i \in I} p_i^{b_i}$, where $b \geq a_i$ for each $i \in I$ which satisfies the following condition.

\[
\forall J \subseteq I : \sigma_z \left( \prod_{i \in J} \prod_{i \in J} p_i^{b_i} \prod_{i \in I \setminus J} p_i^{c_i} \right) = \sigma_z \left( \prod_{i \in J} \prod_{i \in J} p_i^{d_i} \prod_{i \in I \setminus J} p_i^{c_i} \right)
\]

for arbitrary $0 \leq c_i < b, d_i \geq b$ and where $J := I \setminus J$.

**Other known results**

- $\tau_{p,p} = p + 1$.
- $\tau_{2^a,2^{a-1}} = 2^{a+1}$.
- $\tau_{p^a,p^a} \leq p^a + p^{\lceil \frac{a}{2} \rceil} + 1$. 