Online Lower Bounds via Duality

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Theorem 1. *There does not exist a deterministic fractional algorithm with a competitive ratio strictly better than e for the d-dimensional Vector Bin Packing problem, where d is arbitrarily large.*

Theorem 2. *There does not exist a randomized algorithm with a competitive ratio strictly better than e for the Capital Investment problem.*

Theorem 3. *There does not exist a randomized algorithm with a competitive ratio strictly better than $1 - \left(1 - \frac{1}{d}\right)^d$ for the d-bounded online Ad-auctions problem.*

Multidimensional Vector Bin Packing

Existing algorithms: $(1+\epsilon)e$-competitive for small coordinates; e-competitive for splittable

The primal linear program:

$v_1 = (1, 0, 0, \ldots, 0)$, $v_2 = (1, 2, 0, \ldots, 0)$, $v_3 = (1, 1, 3, \ldots, 0)$, …, $v_d = (1, 1, 1, \ldots, d)$.

$x_{i,j}$ – the total fraction of vectors of type v_i assigned to bins opened in phase j $(i \geq j)$.

c – a variable representing the competitive ratio guarantee.

$$
\begin{array}{lll}
\min c & c, x_{i,j} \geq 0 & \forall i \geq j : i, j \in [d] \\
\text{s.t.: } \displaystyle\sum_{r=j}^{d} v_r(k) x_{r,j} \leq c & \text{constraint } z_{k,j} & \forall k, j \in [d] \\
\displaystyle\sum_{r=1}^{i} x_{i,r} = 1 & \text{constraint } y_i & \forall i \in [d]
\end{array}
$$

The dual linear program:

$$
\begin{array}{lll}
\max \displaystyle\sum_{r=1}^{d} y_r & z_{k,j} \geq 0 & \forall k \geq j : k, j \in [d] \\
\text{s.t.: } \displaystyle\sum_{k=1}^{d}\sum_{j=1}^{d} z_{k,j} \leq 1 & \text{constraint } c & \\
y_i \leq i \cdot z_{i,j} + \displaystyle\sum_{r=j}^{i-1} z_{r,j} & \text{constraint } x_{i,j} & \forall i \geq j : i, j \in [d]
\end{array}
$$

Formal feasible dual variables assignment:

$$
y_i = \frac{1}{i}, \quad z_{k,j} = \begin{cases} \frac{1 - \ln(k/j)}{k^2} & \text{if } j \leq k \leq \lfloor e \cdot j \rfloor, \\ 0 & \text{otherwise.} \end{cases}
$$

Online Capital Investment

Existing algorithms:
4-competitive deterministic algorithm (3.618 lower bound)
2.88-competitive randomized algorithm
e-competitive algorithm for slightly less general setting

Sequence definition: n machines where machine m_i has a capital cost of $i+1$ and a production cost of 2^{-i^2}. Introduce $2^{k^2} - 2^{(k-1)^2}$orders for units in phase k (2 units in first), n phases in total

The primal linear program:

- $x_{k,i}$ – the fraction bought of the i^{th} machine in the k^{th} phase.
- $q_{k,i}$ – the fraction of products produced by the i^{th} machine in the k^{th} phase.
- c – a variable representing the competitive ratio guarantee.

$$
\begin{array}{lll}
\min c & c, x_{k,i}, q_{k,i} \geq 0 & \forall k, i \in [n] \\
\text{s.t.: } \sum_{r=1}^{k} x_{r,i} \geq q_{k,i} & \text{constraint } y_{k,i} & \forall k, i \in [n] \\
\sum_{i=1}^{n} q_{k,i} = 1 & \text{constraint } w_k & \forall k \in [n] \\
\sum_{r=1}^{k} \sum_{i=1}^{n} (i+1) \cdot x_{r,i} + 2^{k^2} \sum_{i=1}^{n} 2^{-i^2} q_{k,i} \leq c \cdot (k+2) & \text{constraint } z_k & \forall k \in [n]
\end{array}
$$

The dual linear program:

$$
\begin{array}{lll}
\max \sum_{k=1}^{n} w_k & y_{k,i}, z_k \geq 0 & \forall k, i \in [n] \\
\text{s.t.: } \sum_{k=1}^{n} (k+2) \cdot z_k \leq 1 & \text{constraint } c & \\
(i+1) \sum_{r=k}^{n} z_r \geq \sum_{r=k}^{n} y_{r,i} & \text{constraint } x_{k,i} & \forall k, i \in [n] \\
y_{k,i} \geq w_k - z_k \cdot 2^{k^2 - i^2} & \text{constraint } q_{k,i} & \forall k, i \in [n]
\end{array}
$$

Formal feasible dual variables assignment: Let ϵ be a small constant, the assignment is:

- $y_{k,i} = w_k$, for $k \leq i \leq n$ and 0 otherwise.
- $z_k = \frac{1}{k(k+1)}$, for all $k \leq n$.
- $w_k = e \cdot (1-\epsilon) \ln\left(\frac{k+1}{k}\right)$, for $k \leq n \cdot \epsilon$ and 0 otherwise.