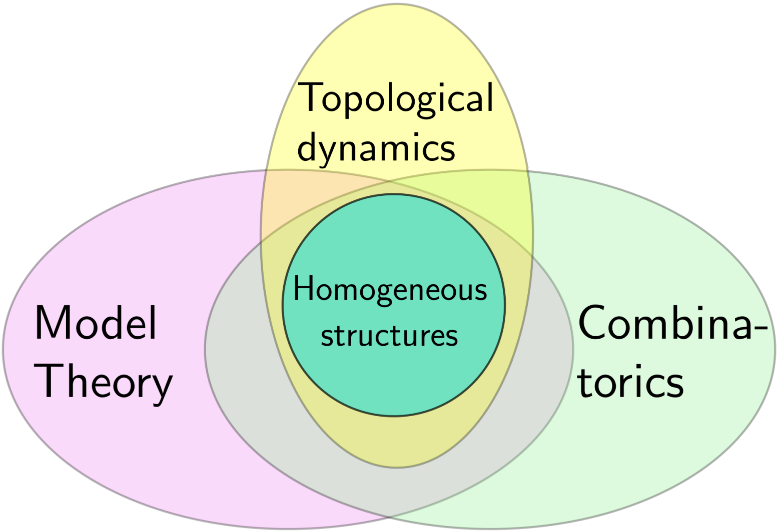
EPPA – context

October 17, 2019

Homogeneous structures



Let $\mathbf{A}$ be a structure (a graph) and let $\mathbf{B}, \mathbf{C}$ be substructures of $\mathbf{A}$ (**induced** subgraphs). If f is an isomorphism $\mathbf{B} \to \mathbf{C}$, we call it a partial automorphism of $\mathbf{A}$.

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- A structure **A** is homogeneous if every partial automorphism of **A** with finite domain extends to an automorphism of **A**.

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Example (Countably infinite homogeneous graphs, Lachlan–Woodrow 1980)

If **G** is a countably infinite homogenous graph, then **G** or its complement $\bar{\mathbf{G}}$ is one of the following:

1. the countable random (Rado) graph,
2. the generic K_n-free graph for $3 \leq n < \infty$,
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1. $(\mathbb{Q}, \leq)$,
2. the countable random k-uniform hypergraph,
3. the countable random tournament,
4. the Urysohn metric space, i.e. the homogeneous complete separable metric space universal for all separable metric spaces.

Definition (EPPA, extension property for partial automorphisms)

Let **B** be a structure (a graph) and let **A** be its substructure (induced subgraph). **B** is an EPPA-witness for **A** if every partial automorphism (isomorphism of induced subgraphs) of **A** extends to an automorphism of **B**.

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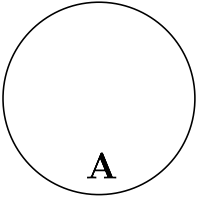

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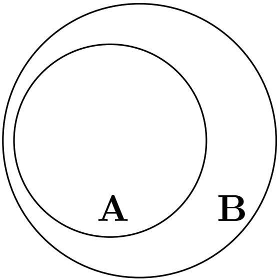

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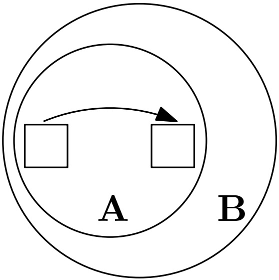

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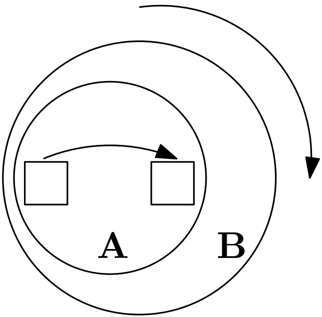

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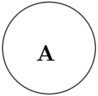
Theorem (Hrushovski, 1992)

The class of all finite graphs has EPPA.

A connection to model theory



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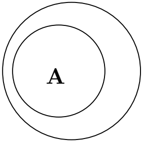

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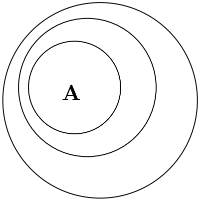

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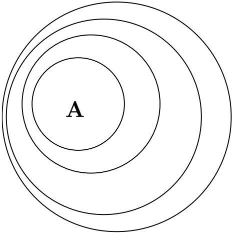

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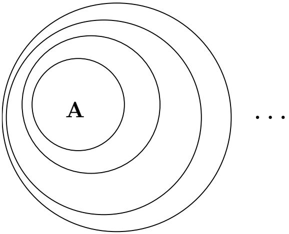

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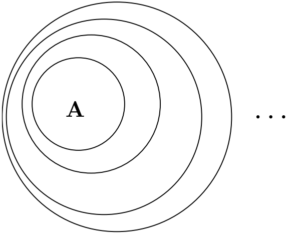

Fact

If $\mathcal{C}$ has EPPA, then it is the class of all finite substructures of a homogeneous structure.

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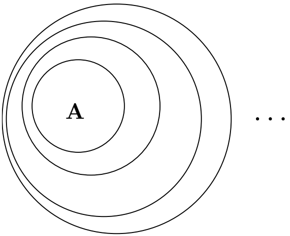

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Remark

EPPA $\iff$ the (topological) automorphism group of the corresponding homogeneous structure can be written as the closure of a chain of proper compact subgroups.

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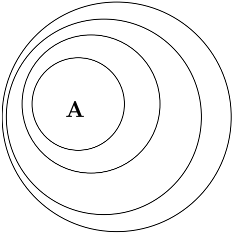

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Moreover, EPPA implies amenability and it is key in proving ample genericity, the small index property etc.

Examples of classes with EPPA

- All finite graphs and K_n-free graphs (Hrushovski 1992, Hodkinson–Otto 2003).
- Finite structures in a relational language (e.g. hypergraphs). (Herwig 1998).
- Metric spaces with distances from $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{N}$ (Solecki 2005, Vershik 2005, Hubička–K–Nešetřil 2018).
- Metric spaces with distances from $S \subseteq \mathbb{R}$ whenever it is possible (Conant 2015, K 2019).
- Metrically homogeneous graphs (Cherlin 2011; AB-WHHKKKP 2017, K 2018).
- Certain classes omitting homomorphisms. (Herwig–Lascar 2000, Hubička–K–Nešetřil 2018).
- Two-graphs (Evans–Hubička–K–Nešetřil 2018).
- n-partite tournaments and semi-generic tournaments (Hubička–Jahel–K–Sabok 2019+).