Handout: EPPA

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Abstract

This is a supplementary material to my seminar talk about the paper [HKN19].

1 Structures

A *relational language* L is a collection of symbols $L = \{R^1, R^2, \ldots\}$ together with associated *arities* $a(R^i) \in \mathbb{N}$. An *L-structure* is a tuple $\mathbf{A} = (A, R^1_{\mathbf{A}}, R^2_{\mathbf{A}}, \ldots)$, where A is the *vertex set* and $R^i_{\mathbf{A}} \subseteq A^{a(R^i)}$ are the *interpretations* of the relations. Usually, we will ignore the difference between a relation and its interpretation. We also adopt the convention that structures are typeset in bold, while their vertex sets are typeset in standard font. Finally, the language is often understood implicitly and we simply talk about structures.

Example.

- Graphs are structures in the language $L = \{E\}$, where E is a binary symbol, whose interpretation is symmetric and irreflexive.
- Linear orders are structures in the language $L = \{\leq\}$, where $\leq$ is again a binary symbol, however, its interpretation is asymmetric, reflexive, transitive and total (i.e. every two vertices are related).

If $\mathbf{A}$ and $\mathbf{B}$ are L-structures and $f\colon A \to B$ is an *injective* function between their vertex sets, we say that f is an *embedding* if for every $R \in L$ and for every $\bar{a} \in A^{a(R)}$, we have

$$\bar{a} \in R_{\mathbf{A}} \iff f(\bar{a}) \in R_{\mathbf{B}},$$

where $f(\bar{a}) = (f(a_1), \ldots, f(a_{a(R)}))$. If f is moreover surjective, we call it an *isomorphism*. And isomorphism $\mathbf{A} \to \mathbf{A}$ is called an *automorphism*. If $A \subseteq B$ and the inclusion is an embedding $\mathbf{A} \to \mathbf{B}$, we say that $\mathbf{A}$ is a *substructure* of $\mathbf{B}$.

A structure $\mathbf{M}$ is *homogeneous* if whenever $\mathbf{A}$ and $\mathbf{B}$ are finite substructures of $\mathbf{M}$ and $f\colon \mathbf{A} \to \mathbf{B}$ is an isomorphism, then there is an automorphism $g\colon \mathbf{M} \to \mathbf{M}$ such that $f \subseteq g$.

Example.

- The countable random graph (the Rado graph) is homogeneous. So are disjoint unions of complete graphs of the same size, or the 5-cycle.
- $(\mathbb{Q}, \leq)$ is homogeneous, while $(\mathbb{Z}, \leq)$ is not.
- There is a classification of homogeneous graphs [LW80], homogeneous directed graphs [Che98] and more.

2 EPPA

Let $\mathbf{A}$ be a finite structure. A *partial automorphism* of $\mathbf{A}$ is an isomorphism $f\colon \mathbf{U} \to \mathbf{V}$ such that $\mathbf{U}$ and $\mathbf{V}$ are substructures of $\mathbf{A}$. Hence, we can understand f as a partial function $A \to A$.

Definition 2.1 (EPPA)**.** Let $\mathbf{A}$ be a finite L-structure. We say that an L-structure $\mathbf{B}$ is an *EPPA-witness* for $\mathbf{A}$ if $\mathbf{A}$ is a substructure of $\mathbf{B}$ and every partial automorphism of $\mathbf{A}$ extends to an automorphism of $\mathbf{B}$.

Let $\mathcal{C}$ be a class of **finite** structures. We say that $\mathcal{C}$ has the *extension property for partial automorphisms* (*EPPA*, also called the *Hrushovski property*) if for every $\mathbf{A} \in \mathcal{C}$ there is $\mathbf{B} \in \mathcal{C}$ which is an EPPA-witness for $\mathbf{A}$

Recall the definition of a homogeneous structure. In the definition of EPPA, $\mathbf{B}$ is intuitively homogeneous for partial automorphisms living within $\mathbf{A}$.

3 Metric spaces

Let L be a set. An L-edge-labelled graph is a graph $\mathbf{G} = (V, E)$ together with a function $d\colon E \to L$ assigning a label to every edge. A metric space is just a complete $\mathbb{R}^{>0}$-edge-labelled graph (we do not represent $d(x, x) = 0$), where some triangles are forbidden (namely those which violate the triangle inequality).

We can also view L-edge-labelled graphs as a relational structures in a binary symmetric language with relations from L. The corresponding maps (and definition of EPPA) generalise naturally, we require them to preserve the labels.

Let $\mathbf{G} = (V, E, d)$ be an L-edge-labelled graph. We will often treat d as a partial function $V^2 \to L$, which is symmetric and defined precisely on E.

Theorem 3.1. *Let L be a finite set and let $\mathbf{A}$ be a finite L-edge-labelled graph. Then there is a finite L-edge-labelled graph $\mathbf{B}$ which is an EPPA-witness for $\mathbf{A}$.*

3.1 Filling-in the missing distances

A *non-metric cycle* is an $\mathbb{R}^{>0}$-edge-labelled cycle with labels $a_0, \dots, a_n$ such that $a_0 > \sum_{i=1}^{n} a_i$.

Proposition 3.2. *Let* $\mathbf{A} = (A, d)$ *be an* $\mathbb{R}^{>0}$*-edge-labelled graph. Then there is* $d' \colon A^2 \to \mathbb{R}^{>0}$ *such that* $d \subseteq d'$ *and* (A, d') *is a metric space if and only if* $\mathbf{A}$ *contains no non-metric cycle as a non-induced subgraph. Moreover, if such a* d' *exists, it can be chosen so that* $\mathrm{Aut}((A, d)) = \mathrm{Aut}((A, d'))$.

Lemma 3.3. *Let* $L \subseteq \mathbb{R}^{>0}$ *be finite, let* $\mathbf{A}$ *be a metric space with distances from* L *and let* $\mathbf{B}$ *be an* L*-edge-labelled graph which is an EPPA-witness for* $\mathbf{A}$*. Suppose that* $\mathbf{B}$ *contains no non-metric cycles on less than* $i \geq 3$ *vertices. Then there is an* L*-edge-labelled graph* $\mathbf{B}'$ *which is an EPPA-witness for* $\mathbf{A}$ *such that* $\mathbf{B}'$ *contains no non-metric cycles on less than* $i + 1$ *vertices.*

This implies the following theorem, which was first proved by Solecki [Sol05].

Theorem 3.4. *The class of all finite metric spaces has EPPA.*

4 Appendix: EPPA for graphs

EPPA is often called the *Hrushovski property*, because Hrushovski was the first to prove this property for the class of graphs [Hru92], upon request from Hodges, Hodkinson, Lascar and Shelah.

Hrushovski's proof of EPPA for graph was group-theoretical and slightly involved. In 2000, Herwig and Lascar [HL00] came up with a very simple combinatorial proof:

Proof of EPPA for graphs. Let $\mathbf{G} = (V, E)$ be a finite graph. First, for convenience, assume that $\mathbf{G}$ is k-regular, that is, every vertex is in k edges.

Define graph $\mathbf{H}$ on vertex set $\binom{E}{k} = \{X \subseteq E : |X| = k\}$ such that $\{X, Y\}$ is an edge of $\mathbf{H}$ if and only if $X \neq Y$ and $X \cap Y \neq \emptyset$ (i.e. $\mathbf{H}$ is the complement of the Kneser graph on k-subsets of E). Define function $\psi \colon V \to \binom{E}{k}$ sending every vertex to the set of edges incident with it ($v \mapsto \{e \in E : v \in e\}$), this is possible since $\mathbf{G}$ is k-regular.

Observe that ψ is an embedding: Indeed, if $uv \in E$, then $\psi(u) \cap \psi(v) = \{uv\} \neq \emptyset$, hence $\psi(u)$ and $\psi(v)$ form an edge of $\mathbf{H}$, analogously if $uv \notin E$. We now show that every partial automorphism of $\psi(\mathbf{G})$ extends to an automorphism of $\mathbf{H}$, hence finishing the proof. Towards this, observe that every permutation of E induces an automorphism of $\mathbf{H}$ by its action of $\binom{E}{k}$.

Let φ be a partial automorphism of $\psi(\mathbf{G})$. Since ψ is an isomorphism, $\psi^{-1}\varphi\psi$ is a partial automorphism of $\mathbf{G}$, so in particular it induces a partial permutation π of E. We aim to extend π to a permutation $\hat{\pi}$ of E so that whenever $X \in \mathrm{Dom}(\varphi)$, then $\hat{\pi}(X) = \varphi(X)$. If we can do it, we are done.

Clearly, if we have $X \neq Y \in \mathrm{Dom}(\varphi)$, then $|X \cap Y| \leq 1$, and if $e \in X \cap Y$, then $e \in \mathrm{Dom}(\pi)$, hence $X \setminus \mathrm{Dom}(\pi)$ and $Y \setminus \mathrm{Dom}(\pi)$ are disjoint. Similarly we get for $X \neq Y \in \mathrm{Range}(\varphi)$ that $X \cap Y \subseteq \mathrm{Range}(\varphi)$. Finally, we shall observe that for every $X \in \mathrm{Dom}(\varphi)$ we have $|X \setminus \mathrm{Dom}(\pi)| = |\varphi(X) \setminus \mathrm{Range}(\pi)|$. This follows from the fact that φ is a partial automorphism: X has the same number of neighbours in what $\mathbf{H}$ induces on $\mathrm{Dom}(\varphi)$ as Y has on what $\mathbf{H}$ induces on $\mathrm{Range}(\varphi)$, and the only edges in $\mathrm{Dom}(\pi)$ are connecting vertices from $\mathrm{Dom}(\varphi)$. Now it follows that we can indeed extend π to $\hat{\pi}$ as desired.

Finally, if $\mathbf{G}$ is not k-regular, then one can "add half-edges" to every vertex to ensure regularity, these half edges will never be in the domain of π. $\square$

References

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