Basic definitions

Definition 1. A *partition* of a graph G is a set $\mathscr{P}$ of non-empty induced supgraphs of G, called *parts*, such that each vertex of G is exactly in one element of $\mathscr{P}$. A partition is *connected* if each part of $\mathscr{P}$ is connected.

Definition 2. The *quotient* of $\mathscr{P}$ is the graph, denoted by $G/\mathscr{P}$, with vertex set $\mathscr{P}$ where $P, Q \in \mathscr{P}$ are adjacent if some vertex in P is adjacent in G to some vertex in Q.

Definition 3. A partition of graph G is *chordal* if it is connected and the quotient $G/\mathscr{P}$ is chordal.

Definition 4. Let $\mathscr{P} = \{P_1, \ldots, P_m\}$ be a partition of a graph G, and let X be an induced subgraph of G. Then the *restriction* of $\mathscr{P}$ to X is the partition of X defined by

$$\mathscr{P}\langle X \rangle := \{G[V(P_i) \cap V(X)]; i \in \{1, \ldots, m\}, V(P_i) \cap V(X) \neq \emptyset\}.$$

Main result

Hadwiger's Conjecture. *Every graph with no K_{t+1}-minor is t-colourable.*

The best known upper bound on the chromatic number for such graph is $\mathcal{O}(t\sqrt{\log t})$.

Question 1 (Reed and Seymour)**.** *Does every graph have a chordal partition such that each part is bipartite?*

This would imply that every graph with no K_{t+1}-minor is $2t$-colourable. However, the following theorem implies the answer is **NO**.

Theorem 2. *For every integer k there is a graph G, such that for every chordal partition $\mathscr{P}$ of G, some part of $\mathscr{P}$ contains K_k.*

Tools

Lemma 3. *Let X be an induced subgraph of a graph G, such that the neighbourhood of each component of $G - V(X)$ is a clique in X. Let $\mathscr{P}$ be connected partition of G with quotient $G/\mathscr{P}$. Then $\mathscr{P}\langle X \rangle$ is a connected partition of X, and the quotient of $\mathscr{P}\langle X \rangle$ is the subgraph of $G/\mathscr{P}$ induced by those parts that intersect X.*

Theorem 2 is a corollary of the following lemma for $r = 1$.

Lemma 4. *For all integers $k \geq 1$ and $r \geq 1$ there is a graph $G(k, r)$ such that for every chordal partition $\mathscr{P}$ of $G(k, r)$ either:*

1. *G contains a K_{kr} subgraph intersecting each of r distinct parts of $\mathscr{P}$ in k vertices, or*
2. *some part of $\mathscr{P}$ contains K_{k+1}.*

Other partitions

Theorem 5. *For every integer k there is a graph G, such that for every perfect partition $\mathscr{P}$ of G, some part of $\mathscr{P}$ contains K_k.*

Theorem 6. *For every integer k and graph H, there is a graph G, such that for every connected partition $\mathscr{P}$ of G, some part of $\mathscr{P}$ contains K_k or the quotient contains H.*

The proofs of these theorems are very similar to the proof of Theorem 2.
