Fast and Compact Exact Distance Oracle for Planar Graphs

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Results

- G is an undirected triangulated planar graph with edge weights and fixed embedding to the plane ($n = |V(G)|$).
- G^* is a dual of G, $d_G(u, v)$ is a distance between u and v in the graph G.
- Distance oracle is a data structure which given any two vertices $u, v \in V(G)$ as a query answers $d_G(u, v)$.
- We assume uniqueness of shortest paths.

Theorem 1. *There exist distance oracles with*

1. $\mathcal{O}(n^2)$ *preprocessing time and* $\mathcal{O}(n^{5/3})$ *space.*
2. $\mathcal{O}(n^{11/6})$ *expected preprocessing time and* $\mathcal{O}(n^{11/6})$ *space.*

Both data structures have the query time $\mathcal{O}(\log n)$.

Theorem 2. *Let S denote the space, P denote the preprocessing time and Q denote the query time. Then, there are distance oracles such that:*

1. $P = \mathcal{O}(n^2), S \geq n^{3/2}$, *and* $Q = \mathcal{O}\left(\frac{n^{5/2}}{S^{3/2}} \log n\right)$.
2. $P = S, S \geq n^{16/11}$, *and* $Q = \mathcal{O}\left(\frac{n^{11/6}}{S^{6/5}} \log n\right)$.

Notions

Definition 3. For a subgraph H of G a vertex $v \in V(H)$ is a *boundary* vertex if there is an edge $\{u, v\} \in E(G) \setminus E(H)$. The set of boundary vertices of H is denoted by $\delta(H)$ and vertices in $V(H) \setminus \delta(H)$ are called *internal vertices.*

Definition 4. A *hole* of a subgraph H of G is a face of H which is not a face of G.

r-division An r-*division* of graph G is a collection $\mathcal{R}(G)$ of subgraphs of G (called *region*) such that

1. Each edge of G is in exactly one region.
2. The number of regions is $\mathcal{O}(n/r)$.
3. Each region contains at most r vertices.
4. Each region has $\mathcal{O}(\sqrt{r})$ boundary vertices.
5. Each region has $\mathcal{O}(1)$ holes.

Fact 5. *An r-division of a graph G can be computed in linear time and space.*

Definition 6. Let $R \in \mathcal{R}(G)$, H be a hole of R, $u \in V(G)$ and $x \in V(H)$. A *Voronoi cell* of x (w.r.t. u, R and H) is a set of vertices $C(x)$ of R such that $v \in C(x)$ iff x is the last vertex of H which are on the shortest path between u and v.

Definition 7. A *Voronoi diagram* of u (w.r.t. R and H) is a collection of all Voronoi cells of u.

Definition 8. Let B^* be a subgraph of R^* consisting of dual boundary edges over all Voronoi cells. We define $\text{Vor}_H(R, u)$ to be a multigraph obtained from B^* by replacing each maximal path whose interior vertices have degree two by a single edge.

- Informally, $\text{Vor}_H(R, u)$ captures the structure of the Voronoi diagram.

Main Idea

1. We compute r-division of G for $r = n^{2/3}$.
2. For each region R we store a distance between any two vertices in R – space $\mathcal{O}(nr) = \mathcal{O}(n^{5/3})$.
3. For each region R and each vertex $u \in V(G)$ we store a distance between u and each boundary vertex of R – space $\mathcal{O}(n^2/\sqrt{r}) = \mathcal{O}(n^{5/3})$.
4. For answering the query $\{u, v\}$:
   (a) If u and v is in the same region or at least one of them is a boundary vertex of some region, then the query is trivial.
   (b) Otherwise, we need to find a boundary vertex w_H of a hole H such that v is in the Voronoi cell of w.
      - It holds that $d_G(u, v) = d_G(u, w_H) + d_R(w_H, u)$ for some hole H of R.
      - For a finding such w we store a recursive decomposition of $\text{Vor}_H(R, u)$ which allow us to perform a binary search. Thus, we can find w in time $\mathcal{O}(\log n)$.