Approximating the rectilinear crossing number

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1 Main result

Drawing of a graph $G \Rightarrow$ crossing number of G, notation $\mathrm{cr}(G)$. Straight-line drawing of $G \Rightarrow$ rectilinear crossing number of G, notation $\overline{cr}(G)$.

Theorem 1 *There is a deterministic $n^{2+o(1)}$-time algorithm for constructing a straight-line drawing of any n-vertex graph G in the plane with*

$$\overline{cr}(G) + O(n^4/(\log\log n)^\delta)$$

crossing pairs of edges, where $\delta > 0$ is an absolute constant.

Corollary 1 *There is a deterministic $n^{2+o(1)}$-time algorithm for constructing a straight line drawing of an n-vertex graph G with $|E(G)| \geq \varepsilon n^2$, where $\varepsilon > 0$ is fixed, such that the drawing has at most $(1+o(1))\overline{cr}(G)$ crossing pairs of edges.*

2 Plane arrangements

Let the *order type* of the set of points $V = \{v_1, v_2, \dots, v_n\}$ be the mapping $\chi : \binom{V}{3} \to \{+1, -1\}$ assigning each triple of V its orientation. We call all vectors $\chi^* \in \{-1,+1\}^{\binom{n}{3}}$ *abstract order types* and say that χ^* is *realizable* if there is a set of n points in plane whose order realizes χ^*.

Given k disjoint sets $V_1, V_2, \dots, V_k$, a *transversal* of $(V_1, \dots, V_k)$ is any k-element sequence $(v_1, v_2, \dots, v_k)$ with $v_i \in V_i$. Sets $V_1, V_2, \dots, V_k$ has *same-type transversals* if all of its transversals has the same order type. A partition of a finite set V is *equitable* if all its parts differ in size by at most one.

Theorem 2 *There is an absolute constant C such that the following holds. For each $0 < \varepsilon < 1$ and for any finite point set V in $\mathbb{R}^2$ there is an equitable partition $V = V_1 \cup V_2 \cup \dots \cup V_K$, with $1/\varepsilon < K < \varepsilon^{-C}$, such that all but at most $\varepsilon\binom{K}{4}$ quadruples of parts $\{V_{i_1}, V_{i_2}, V_{i_3}, V_{i_4}\}$ have same-type transversals.*

Lemma 1 *Given a graph G on K vertices, we can find a straight-line drawing of G with $\overline{cr}(G)$ pairs of crossing edges in $2^{O(K^3)}$ time.*

Lemma 2 *Let G be an edge weighted graph G on K vertices where the weight of each edge uses at most B bits. Then we can find a straight-line drawing of G with $\overline{cr}(G)$ weighted edge crossings in $2^{O(K^3)}B^2$ time.*

3 Frieze–Kannan regularity lemma

Let G be an edge weighted graph with weights in $[0,1]$ For $S, T \subset V(G)$ we define

$$e_G(S,T) = \sum_{u \in S, v \in T} w_G(uv).$$

Let G and G' be two graphs on the same vertex set V. The *cut-distance* between G and G' is defined by

$$d(G,G') = \max_{S,T \subset V} |e_G(S,T) - e_{G'}(S,T)|.$$

Generalization of crossing number to weighted graphs. Let G be an edge weighted graph G, $\mathcal{D}$ its straight line drawing and X the set of pairs of crossing edges in $\mathcal{D}$. The *rectilinear crossing number* of G is defined by

$$\overline{cr}(G) = \min_{\mathcal{D}} \sum_{(uv,st)\in X} w_G(uv)w_G(s,t).$$

Let $\varepsilon > 0$ and $G = (V, E)$ be a graph on n vertices. An equitable partition $\mathcal{P} : V = V_1 \cup V_2 \cup \cdots \cup V_K$ is said to be *ε-Frieze-Kannan* if for all $S, T \subset V$ we have

$$\left| e_G(S,T) - \sum_{1\le i,j\le K} e_G(V_i,V_j)\frac{|V_i \cap S||V_j \cap T|}{|V_i||V_j|} \right| < \varepsilon n^2.$$

Theorem 3 *There is an absolute constant c such that the following holds. For each $\varepsilon > 0$ and for any graph $G = (V, E)$ on n vertices, there is a deterministic algorithm which finds ε-Frieze-Kannan-regular partition on V with at most $2^{\varepsilon^{-c}}$ parts which runs in $2^{2^{\varepsilon^{-c}}} n^2$ time.*

Let $G = (V, E)$ and $\mathcal{P} : V = V_1 \cup V_2 \cup \cdots \cup V_K$ be an *ε-Frieze-Kannan* regular partition of V from previous theorem. Let $G/\mathcal{P}$ be a wighted graph on the set $\{1, 2, \ldots, K\}$ with weights

$$w_{G/\mathcal{P}}(ij) = \frac{e_G(V_i, V_j)}{(n/K)^2} \quad 1 \le i \ne j \le K,$$

and $G_{\mathcal{P}}$ be a weighted graph on V with weights

$$w_{G_{\mathcal{P}}}(uv) = \begin{cases} \frac{e_G(V_i,V_j}{(n/K)^2} & \text{for } u \in V_i, v \in V_j, 1 \le i \ne j \le K, \\ 0 & \text{for } u, v \in V_i, 1 \le i \le K. \end{cases}$$

Lemma 3 *Let $\varepsilon \in (0, 1/2)$ and let G and G' be two n-vertex edge-wighted graphs on the same vertex set V. If $d(G, G') < \varepsilon n^2$ then*

$$|\overline{cr}(G) - \overline{cr}(G')| \le \varepsilon^{1/4C} n^4,$$

where C is an absolute constant from Theorem 2.

The *blow-up* graph $G[m]$ of an edge weighted graph G on the set $\{1, 2, \ldots, K\}$ is a graph obtained from G by replacing each vertex i by an independent set U_i of order m, where each weight between U_i and U_j has weight $w_G(ij)$, $i \ne j$.

Lemma 4 *Let G be a graph and $G[m]$ be its blow-up. Then*

$$0 \le \overline{cr}(G[M]) - m^4\overline{cr}(G) \le K^3 m^4.$$

4 The algorithm

1. Take any G on n vertices. Set $\varepsilon = (\log\log n)^{-1/2c}$, where c is a constant from Theorem 3. We apply Theorem 3 and obtain an equitable partition $\mathcal{P} : V = V_1 \cup V_2 \cup \cdots \cup V_K$ such that $1/\varepsilon < K < 2^{\sqrt{\log\log n}}$. That can be done in $n^{2+o(1)}$.

2. We consider edge wighted graph $G/\mathcal{P}$ on $[K]$ such that $w_{G/\mathcal{P}}(ij) = \frac{e_G(V_i,V_j)}{(n/K)^2}$. Then algorithm from Lemma 2 finds a drawing of $G/\mathcal{P}$ with crossing number $\overline{cr}(G/\mathcal{P})$. Let U be the point set for such drawing. This can be done in $2^{O(K^3)} = n^{o(1)})$ time.

3. We draw $G = (V, E)$. Let L be the set of lines spanned by U and δ minimal distance of L and U. Such δ uses at most $2^{K\log K}$ bits. Set $D_i = (v_i, \delta/10)$ disc around v_i and choose points of V_i in D_i such that the point set V is in general position. Then any quadruple of points from different V_i's has same type transversals.

4. Finally we draw all edges of G in $O(n^2)$ time and return the drawing of G.