Three conjectures in extremal spectral graph theory (Tait, Tobin)

Intro

Observation 1. *Let A be an adjacency matrix of a graph G with eigenvalues $\lambda_1 \geq \lambda_2 \geq, \ldots, \lambda_n$. Let v be an eigenvector corresponding to λ_1 scaled such that the maximum entry is equal to 1. Let $x \in V(G)$ be an arbitrary vertex such that $v_x = 1$.*

$$\lambda_1 \mathbf{v}_u = \sum_{w \sim u} \mathbf{v}_w \tag{1}$$

Equation 1 applied to $u = x$:

$$\lambda_1 = \sum_{y \sim x} \mathbf{v}_y \tag{2}$$

$$\lambda_1^2 = \sum_{y\sim x}\sum_{z\sim y} \mathbf{v}_z = \sum_{y\sim x} \sum_{\substack{z\sim y \\ z\in N(x)}} \mathbf{v}_z + \sum_{y\sim x} \sum_{\substack{z\sim y \\ z\notin N(x)}} \mathbf{v}_z \leq 2e\left(N(x)\right) + e\left(N(x), V(G)\setminus N(x)\right) \tag{3}$$

Observation 2.*[Rayleigh quotient characterization of λ_1]*

$$\lambda_1 = \max_{\boldsymbol{z}\neq \boldsymbol{0}} \frac{\boldsymbol{z}^T A \boldsymbol{z}}{\boldsymbol{z}^T \boldsymbol{z}}$$

Theorem 1.*[Mantel's Theorem] Let G be a triangle-free graph on n vertices. Then G contains at most $\lfloor n^2/4 \rfloor$ edges. Equality occurs if and only if $G = K_{\lfloor n/2\rfloor \lceil n/2 \rceil}$.*

Theorem 2.*[Stanley's Bound] Let G be a graph with m edges. Then*

$$\lambda_1 \leq \frac{1}{2}\left(-1 + \sqrt{1+8m}\right).$$

Equality occurs iff G is a clique and isolated vertices.

Outerplanar graphs of maximum spectral radius

Definition 1. Spectral radius of a square matrix is $\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$.

Theorem 7. *The outerplanar graph on n vertices of maximum spectral radius is one vertex connected with every vertex of the path P_{n-1}, that is $K_1 + P_{n-1}$ where $+$ represents the graph join operation.*

Lemma 3. $\lambda_1 > \sqrt{n-1}$.

Lemma 4. *For any vertex u, we have $d_u > \mathbf{v}_u n - 11\sqrt{n}$.*

Lemma 5. *We have $d_x > n - 11\sqrt{n}$ and for every other vertex u, $\mathbf{v}_u < 23/\sqrt{n}$ for sufficiently large n (let $C_1 = 23$).*

Lemma 6. *Let $B = V(G) \setminus (N(x) \cup \{x\})$. Then*

$$\sum_{z\in B} \mathbf{v}_z < C_2/\sqrt{n}$$

for some constant C_2.