On the number of monotone sequences

by *Wojciech Samotij and Benny Sudakov*

presented by Martin Balko

The problem:

- We call a sequence σ of n numbers an *n-sequence*. We assume that σ is a permutation of $[n]$, i.e., $\sigma \in S_n$.

  Problem. *Determine the minimum number of* monotone *(that is, monotonically increasing or monotonically decreasing) subsequences of length $k+1$ in an n-sequence.*

  Theorem 1 (The Erdős–Szekeres Theorem, 1935)**.** *For every $k, n \in \mathbb{N}$, every n-sequence contains at least $n - k^2$ monotone subsequences of length $k+1$.*

- Let $\tau_{k,n}$ be a sequence of k increasing sequences of length $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ that are concatenated in decreasing order.

- For $\sigma \in S_n$, let $m_k(n)$ be the number of monotone subsequences of length $k+1$ in σ and let $m_k(n) := \min\{m_k(\sigma) \colon \sigma \in S_n\}$. Let $r_{k,n}$ be the unique number $r \in \{0, \dots, k-1\}$ satisfying $r \equiv n (\mathrm{mod} k)$.

  Conjecture 2 (Myers, 2002–2003)**.** *For all k and n,*

$$m_k(n) = m_k(\tau_{k,n}) = r_{k,n}\binom{\lceil n/k \rceil}{k+1} + (k - r_{k,n})\binom{\lfloor n/k \rfloor}{k+1}.$$

Main result:

- The conjecture of Myers is true for all sufficiently large k, as long as n is not much larger than k^2.

  Theorem 3. *There exist an integer k_0 and a number $c \in \mathbb{R}^+$ such that $m_k(n) = m_k(\tau_{k,n})$ for all k and n satisfying $k \geq k_0$ and $n \leq k^2 + ck^{3/2}/\log k$. Moreover, if $n \neq k^2 + k + 1$ and $m_k(\sigma) = m_k(n)$ for some $\sigma \in S_n$, then σ contains monotone subsequences of length $k+1$ of only one type (increasing or decreasing).*

- Surprisingly, if $n = k^2 + k + 1$, then there are $\sigma \in S_n$ with $m_k(\sigma) = m_k(n) = 2k + 1$ which contain both increasing and decreasing subsequences of length $k+1$.

Reformulation of the main result:

- Every $\sigma \in S_n$ admits a natural representation as a poset $P_\sigma = ([n], \leq_\sigma)$ in which its increasing and decreasing subsequences are mapped to chains and antichains, respectively, of the same length.

- A set A of elements of a poset is *homogenous* if A is a chain or an antichain.

- Given a poset P, let $h_k(P)$ be the number of homogenous $(k+1)$-element sets in P and let $h_k(n) := \min\{h_k(P) \colon P \text{ is a poset with } n \text{ elements}\}$.

  Problem. *For every k and n, determine the minimum number of homogenous $(k+1)$-element sets in a poset with n elements. In particular, is it true that $h_k(n) = m_k(n)$ for all k and n?*

- For a poset P of *order dimension* at most two (that is, P is the intersection of two linear orders), a *dual poset* P^* is a poset on $[n]$ such that every pair of elements is comparable in either P or P^* but not both of them.

  Theorem 4. *There exist an integer k_0 and $c \in \mathbb{R}^+$ such that the following is true. Let k and n be integers satisfying $k \geq k_0$ and $n \leq k^2 + ck^{3/2}/\log k$. If P is an n-element poset of order dimension at most two, then*

$$h_k(P) \geq m_k(\tau_{k,n}).$$

  Moreover, if $h_k(P) = m_k(\tau_{k,n})$ and $n \neq k^2 + k + 1$, then P can be decomposed into k chains or k antichains of length $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ each.

  If $h_k(P) = m_k(\tau_{k,n})$ and $n = k^2 + k + 1$, then P (or P^*) can additionally belong to one of two families of n-element posets with exactly $2k+1$ homogenous $(k+1)$-sets that contain both chains and antichains with $k+1$ elements.

Some notation:

- Let $(P, \leq)$ be a poset. The *height* $h(P)$ and the *width* $w(P)$ of P are the cardinalities of the largest chain and the largest antichain in P, respectively.
- For every positive integer i, let $A_i := \{x \in P\colon \text{the longest chain } L \text{ with } \max L = x \text{ has } i \text{ elements}\}$.
- Let G_i be the bipartite graph on the vertex set $A_i \cup A_{i+1}$ whose edges are all pairs xy with $x \in A_i$ and $y \in A_{i+1}$ such that $x \leq y$.
- For $i \in [h(P)]$ and $x \in A_i$, let $u_i(x)$ be the number of chains $L \subseteq P$ of length $h - i + 1$ with $\min L = x$.
- We define $A'_i := \{x \in A_i\colon u_i(x) \geq 1\}$, $\Sigma_i := \sum_{x \in A_i} u_i(x)$, and $B_{i+1} := \{y \in A'_{i+1}\colon \deg_{G_i}(y) = 1\}$.
- The *k-surplus* $s_k(P)$ of P is defined by $s_k(P) := n - h(P)k$. It measures the distance between a poset P and a union of k chains.

Outline of the proof of Theorem 4:

- We proceed by induction on n tacitly assuming $h(P) \geq w(P)$.
- Each $x \in P$ that is contained in at least $m_k(\tau_{k,n}) - m_k(\tau_{k,n-1})$ homogenous $(k+1)$-sets can be removed.
- We first show that if P is 'far' from being a union of k chains (or k antichains), then $m_k(P)$ is much larger than $m_k(\tau_{k,n})$ (Corollary 7).
- We prove a sequence of lower bounds on Σ_1. By Lemma 8, for each i such that $A_i \cup A_{i+1}$ contains an antichain of length $k+1$ either $\Sigma_i - \Sigma_{i+1}$ is large or $A_i \cup A_{i+1}$ contains many $(k+1)$-element antichains. Each of these situations implies $h_k(P) > m_k(\tau_{k,n})$. Here, Corollary 10 translates lower bounds on Σ_1 to lower bounds on $h_k(P)$. The proof of each of the bounds on Σ_1 relies on the analysis of the graphs G_i.
- If P does not satisfy any of these conditions, then P becomes greatly restricted. A careful case analysis then shows that $h_k(P) \geq m_k(\tau_{k,n})$ and this is strict unless $n = k^2 + k + 1$ and P (or P^*) belongs to one of the two special families of posets.

Lemma 5. *Suppose that $a \geq b > 0$, let $\mathcal{F}$ be an arbitrary family of a-element sets, and define*

$$\partial_b \mathcal{F} := \{B\colon |B| = b \text{ and } B \subseteq A \text{ for some } A \in \mathcal{F}\}.$$

Then $|\partial_b \mathcal{F}| \geq \min\{|\mathcal{F}|/2, 2^b\}$.

Lemma 6. *Let d, k, and s be integers satisfying $1 \leq d \leq k$ and suppose that P is a poset such that $s_k(P) \geq s$ and deletion of no $s/2$ elements reduces the height of P. Then P contains either at least 2^d antichains with $k+1$ elements or at least $2^{\lfloor s/(2d) \rfloor}$ chains of length $h(P)$.*

Corollary 7. *Let k and t be integers satisfying $0 < t \leq k/2$ and suppose that P is a poset of order dimension at most two such that $h(P) \geq w(P)$ and $s_k(P) \geq 3t$. Then P contains at least $2^{\sqrt{t}-1}$ homogenous $(k+1)$-sets.*

Lemma 8 (Key lemma)**.** *Let $\ell := \lceil n/k \rceil - k - 1$ and $F := \{i \in [k+\ell]\colon |A_i| \geq k+1\}$. If $i \in F \cap [k+\ell-1]$, then $A_i \cup B_{i+1}$ contains at least $2^{\min\{k, |B_{i+1}|\}}$ antichains with $k+1$ elements and*

$$\Sigma_i \geq \Sigma_{i+1} + \sum_{y \in A'_{i+1} \setminus B_{i+1}} u_{i+1}(y) \geq \Sigma_{i+1} + |A'_{i+1}| - |B_{i+1}|.$$

Lemma 9. *Suppose that M is a positive integer, X and Y are arbitrary sets, and $f_1, \ldots, f_M\colon X \to Y$ are pairwise different functions. There exist sets $X_1, \ldots, X_M \subseteq X$ with $|X_i| \leq \log_2 M$ for all $i \in [M]$ such that*

$$f_i \restriction_{X_i \cup X_j} \neq f_j \restriction_{X_i \cup X_j} \qquad \text{for all } i \neq j.$$

Corollary 10. *Let k, ℓ, and M be positive integers, let P be a poset of height $k+\ell$, and suppose that $m := \log_2 M + 1 \leq k/4$.*

(i) If P contains at least M chains of length $k+\ell$, then it contains at least

$$\exp\left(-\frac{2(\ell-1)m}{k}\right) \cdot M \binom{k+\ell}{k+1}$$

chains of length $k+1$.

(ii) Given any $y \in P$, (i) still holds if we replace 'chains' with 'chains containing y'.