Large subgraphs without short cycles

F. Foucaud, M. Krivelevich & G. Perarnau
presented by Vojtěch Kaluža

May 12, 2016

Preliminary notation and definitions.

- The letters F, G, H and Γ always denote a simple undirected graph.
- We say that a graph G is *F-free* if there is no subgraph of G isomorphic to F.
- *Turán graphs* and *Turán numbers* - Let $\mathcal{F}$ be a class of graphs. We write $\mathrm{Ex}(G, \mathcal{F})$ for a largest subgraph of G that is F-free for every $F \in \mathcal{F}$. We denote by $\mathrm{ex}(G, \mathcal{F})$ the number of edges of $\mathrm{Ex}(G, \mathcal{F})$.
- $f(m, \mathcal{F}) = \min\{\mathrm{ex}(G, \mathcal{F})\colon\ |E(G)| = m\}$.
- $g(G)$ stands for the girth of G, $\delta(G)$ for its minimum degree and $\Delta(G)$ for its maximum degree.
- $d(G, \mathcal{F}) = \max\{\delta(H)\colon V(H) = V(G)$ and H is $\mathcal{F}$-free subgraph of $G\}$.
- $h(\delta, \Delta, \mathcal{F}) = \min\{d(G, \mathcal{F})\colon \delta(G) = \delta$ and $\Delta(G) = \Delta\}$.
- $\chi(F)$ denotes the chromatic number of F and $\chi(\mathcal{F}) = \min_{F \in \mathcal{F}} \chi(\mathcal{F})$.
- $\mathcal{F}_r = \{C_3, C_4, \ldots, C_{2r}, C_{2r+1}\}$ and $\mathcal{F}_r^{\mathrm{even}} = \{C_4, C_6, \ldots, C_{2r}\}$.

Results.

Proposition 1: *For every graph G and for every $k \geq 3$ there exists a $(k-1)$–partite subgraph H of G such that for every $v \in V(G)$,*

$$d_H(v) \geq \left(1 - \frac{1}{k-1}\right) d_G(v) .$$

Moreover, for every $\mathcal{F}$ with $\chi(\mathcal{F}) = k$, we have

$$f(m, \mathcal{F}) = \left(1 - \frac{1}{k-1}\right) m + o(m)$$

and

$$h(\delta, \Delta, \mathcal{F}) = \left(1 - \frac{1}{k-1}\right) \delta + o(\delta) .$$

Theorem 2: *For every $r \geq 2$ there exists $c = c(r) > 0$ such that for every $m \geq m_0$,*

$$f(m, r) := f(m, \mathcal{F}_r^{even}) \geq \frac{c}{\log m} \min_{k|m} \mathrm{ex}(K_{k,m/k}, \mathcal{F}_r^{\mathrm{even}}) .$$

Corollary 3: *There exists a constant $c > 0$ such that for every $m \geq m_0$,*

$$f(m, 2) \geq \frac{cm^{2/3}}{\log m} .$$

Theorem 4: *Let G be a graph with minimum degree δ and (large enough) maximum degree Δ such that $\mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta \geq \alpha\Delta^2 \log^4 \Delta$ for some large constant $\alpha > 0$. Then, for every $r \geq 2$ there exists a spanning subgraph H of G with $g(H) \geq 2r + 2$ and $\delta(H) \geq \frac{c \cdot \mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta}{\Delta^2 \log \Delta}$, for some small constant $c > 0$. In particular, under the above conditions on δ and Δ,*

$$h(\delta, \Delta, r) \geq h(\delta, \Delta, \mathcal{F}_r) \geq \frac{c \cdot \mathrm{ex}(K_\Delta, \mathcal{F}_r)\delta}{\Delta^2 \log \Delta} .$$

Corollary 5: *For every Δ and δ such that* $\mathrm{ex}(K_\Delta,\mathcal{F}_r)\delta \geq \alpha\Delta^2\log^4\Delta$ *for some large constant $\alpha > 0$, we have*

$$h(\delta,\Delta,r) \geq h(\delta,\Delta,\mathcal{F}_r) = \Omega\left(\frac{\delta}{\Delta^{1-\frac{2}{3r-2}}\log\Delta}\right) .$$

Proofs.

Definition: *For every graph G, every graph Γ with $V(\Gamma) = [\ell]$ and every vertex labeling $\chi : V(G) \to [\ell]$ we define the spanning subgraph $H'_{(\chi,\Gamma)} \subseteq G$ as the subgraph with vertex set $V(G)$ where an edge $e = uv$ is present if and only if $uv \in E(G)$ and $\chi(u)\chi(v) \in E(\Gamma)$.*

Definition: *For every graph G, every graph Γ with $V(\Gamma) = [\ell]$ and every vertex labeling $\chi : V(G) \to [\ell]$ we define the spanning subgraph $H^*_{(\chi,\Gamma)} \subseteq G$ as the subgraph with vertex set $V(G)$ such that an edge $e = uv$ is present in H^* if all the following properties are satisfied:*

1. *$uv \in E(G)$ and $\chi(u)\chi(v) \in E(\Gamma)$, that is $e \in E(H')$,*
2. *for every $w \neq v$, $w \in N_G(u)$, we have $\chi(w) \neq \chi(v)$, and*
3. *for every $w \neq u$, $w \in N_G(v)$, we have $\chi(w) \neq \chi(u)$.*

- We say that a colouring $\chi\colon V(G) \to [l]$ is *t-frugal* if for every $v \in V(G)$ and every colour $c \in [l]$ we have $\left|N_G(v) \cap \chi^{-1}(c)\right| \leq t$; that is, v is a neighbour of at most t vertices of the same colour.
- A cycle in a coloured graph G is called *rainbow* if all its vertices have distinct colour.
  A path in G is called *inner-rainbow* if its both endpoints have the same colour c and all other vertices of the path are coloured with distinct colours different from c.

Lemma 13: *Let G be a graph with maximum degree Δ and minimum degree δ that admits a t–frugal coloring χ without rainbow cycles of length at most $2r+1$ and maximal inner-rainbow paths of length l for every $3 \leq l \leq 2r$. If $\delta > 129t^3\log\Delta$ and Δ is large enough, then there exists a subgraph $H \subseteq G$ such that*

1. *$\forall v \in V(G)$, $d_H(v) \geq \frac{d_G(v)}{4t}$, and*
2. *$g(H) \geq 2r+2$.*

Probabilistic tools.

Lemma (Chernoff inequality for binomial distributions): *Let $X \sim \mathrm{Bin}(N,p)$ be a Binomial random variable. Then for all $0 < \varepsilon < 1$,*

$$\Pr(X \leq (1-\varepsilon)Np) < \exp\left(-\frac{\varepsilon^2}{2}Np\right).$$

Lemma (Azuma inequality): *Let $L\colon S^T \to \mathbb{R}$ be a functional such that for every g and g' differing in just one coordinate from the product space S^T, we have $|L(g) - L(g')| \leq 1$. Let $|T| = l$. Then for all $\lambda > 0$*

1. $\Pr(L \leq \mathbb{E}(L) - \lambda\sqrt{l}) < e^{-\frac{\lambda^2}{2}}$,
2. $\Pr(L \geq \mathbb{E}(L) + \lambda\sqrt{l}) < e^{-\frac{\lambda^2}{2}}$.

Lemma (Weighted Lovász Local Lemma): *Let $\mathcal{A} = \{A_1,\ldots,A_N\}$ be a set of events and let H be a dependency graph for $\mathcal{A}$. If there exist weights $w_1,\ldots,w_N \geq 1$ and a real $p \leq \frac{1}{4}$ such that for each $i \in [N]$:*

1. $\Pr(A_i) \leq p^{w_i}$, *and*
2. $\sum_{j\colon ij\in E(H)}(2p)^{w_j} \leq \frac{w_i}{2}$,

then

$$\Pr\left(\bigcap_{i=1}^{N}\overline{A_i}\right) > 0\,.$$