Deterministic Communication vs. Partition Number

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Let $F : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ be a function and $M_F \in \{0,1\}^{\mathcal{X}\times\mathcal{Y}}$ be a matrix, describing the function F (i.e. $M_{x,y} = F(x,y)$). Let $\chi_i(F)$ be the minimal number of rectangles needed to partition the set of all i-entries in M_F.

Definition 1. The *partition number* of F is defined by

$$\chi(F) = \chi_0(F) + \chi_1(F).$$

Notation Let F be a Boolean function. We use $\mathsf{P}^{\mathsf{cc}}(F)$ for a deterministic communication complexity of F and $\mathsf{P}^{\mathsf{dt}}(F)$ for a decision tree complexity of F. Similarly, we use $\mathsf{UP}^{\mathsf{cc}}(F), \mathsf{UP}^{\mathsf{dt}}(F)$ for an unambiguous nondeterministic communication/decision tree complexity of F. Let $co\mathsf{UP}^{\mathsf{dt}}(F) = \mathsf{UP}^{\mathsf{dt}}(\neg F)$ and $2\mathsf{UP}^{\mathsf{dt}}(F) = \max\{\mathsf{UP}^{\mathsf{dt}}(F), co\mathsf{UP}^{\mathsf{dt}}(F)\}$. We define $\mathsf{UP}^{\mathsf{cc}}(F) = \lceil \log \chi_1(F) \rceil$, $co\mathsf{UP}^{\mathsf{cc}}(F) = \lceil \log \chi_0(F) \rceil$ and $2\mathsf{UP}^{\mathsf{cc}} = \lceil \log \chi(F) \rceil$ which is in

$$\max\{\mathsf{UP}^{\mathsf{dt}}(F), co\mathsf{UP}^{\mathsf{dt}}(F)\} \pm \mathcal{O}(1)$$

Theorem 2. *1. There is an F such that $\mathsf{P}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(\log^2 \chi_1(F))$.*

2. There is an F such that $\mathsf{P}^{\mathsf{cc}}(F) \geq \tilde{\Omega}(\log^{1.5} \chi(F))$.

The main theorem which we prove is a simulation theorem which tells us how to simulate decision tree in communication and vice versa.

Theorem 3 (Simulation Theorem)**.** *There is a gadget $g : \mathcal{X} \times \mathcal{Y} \to \{0,1\}$ where the size of Alice's input is $\log|\mathcal{X}| = \Theta(\log n)$ bits such that for all functions $f : \{0,1\}^n \to \{0,1\}$ we have*

$$\mathsf{P}^{\mathsf{cc}}(f \circ g^n) = \mathsf{P}^{\mathsf{dt}}(f) \cdot \Theta(\log n)$$

The gadget in the simulation theorem is only an index function $g : [m] \times \{0,1\}^m \to \{0,1\}$ where $m = poly(n)$ and $g(x,y) = y_x$. The simulation theorem allows us to convert communication problems to decision tree problems and vice versa. Moreover, one inequality of the simulation theorem is very easy and also holds for unambiguous measures too. The following inequality holds for $\mathcal{C} \in \{\mathsf{P}, \mathsf{UP}, 2\mathsf{UP}\}$:

$$\mathcal{C}^{\mathsf{cc}}(f \circ g^n) \leq \mathcal{C}^{\mathsf{dt}}(f) \cdot \mathcal{O}(\log n).$$

Thus, it suffice to prove the following theorem (which is quite easy to prove) for proving of Theorem 2.

Theorem 4. *1. There is an F such that $\mathsf{P}^{\mathsf{dt}}(F) \geq \tilde{\Omega}(\mathsf{UP}^{\mathsf{dt}}(F)^2)$.*

2. There is an F such that $\mathsf{P}^{\mathsf{dt}}(F) \geq \tilde{\Omega}(2\mathsf{UP}^{\mathsf{dt}}(F)^{1.5})$.

Simulation algorithm

The simulation algorithm uses communication protocol for $F = f \circ g^N$ for computing the function f on input $z = (z_1, \ldots, z_N) \in \{0,1\}^N$ making at most $\mathsf{P}^{\mathsf{cc}}(F)/\Theta(\log m)$ queries to z, where $m = N^{20}$.

Notation For two sets $A \subseteq [m]^n, B \subseteq (\{0,1\}^m)^n$ (subset of inputs for F) we use the following definitions.

- Assuming A and B are not empty, let $\alpha(A)$ be such that $|A| = 2^{-\alpha(A)}|[m]^n|$ and let $\beta(B)$ be such that $|B| = 2^{-\beta(B)}|(\{0,1\}^m)^n|$.
- Let $I \subseteq [n]$. Then, $A_I = \big\{(x_i)_{(i\in I)} : (x_1,\dots,x_n) \in A\big\}$. Analogously we define B_I. Example: $A = \big\{(1,2,3),(3,2,1)\big\}$, then $A_{\{1,3\}} = \big\{(1,3),(3,1)\big\}$.
- Let $U \subseteq [m], i \in [n]$. Then, $A^{i,U} = \{x \in A : x_i \in U\}$. Analogously we define $B^{i,V}$ for $V \subseteq \{0,1\}^m$. Example: $A = \big\{(1,2,3),(3,2,1)\big\}, U = \{1,2\}, i = 1$, then $A^{i,U} = \big\{(1,2,3)\big\}$.
- Let $i \in [n]$. Then, graph $G_i(A)$ is a bipartite graph where partite sets are the sets $[m]$ and $A_{[n]\setminus\{i\}}$. There is an edge between j and $(x_1,\dots,x_{i-1},x_{i+1},\dots,x_n)$ if
$$(x_1,\dots,x_{i-1},j,x_{i+1},\dots,x_n) \in A.$$
We denote the average degree and minimum degree of vertices of $A_{[n]\setminus\{i\}}$ by $\mathrm{avg}_i(A)$, $\delta_i(A)$ respectively.
- We say A is *thick* if $\mathrm{avg}_i(A) \geq m^{17/20}$ for all $i \in [n]$.

Simulation The simulation proceeds in iterations where in each iteration the algorithm descends in the communication tree (communication step) or make a query to the input z (query step). When the algorithm reach the leaf of the communication tree, it ends and outputs the answer—the value of the leaf. The algorithm maintains 3 sets $A \subseteq [m]^N, B \subseteq (\{0,1\}^m)^N, I \subseteq [N]$, at the beginning we have $A = [m]^N, B = (\{0,1\}^m)^N$ and $I = [N]$. There are unqueried indices in the set I. The following invariants will hold for A and B after every iteration:

1. A_I is thick—technical invariant.
2. $A \times B \subseteq R_v$, where R_v is a rectangle corresponding to the current node in the communication tree.
3. $g(x_i, y_i) = z_i$ for every queried index i (i.e. $i \in [N] \setminus I$).

We use $\alpha(A_I)$ as potential which guarantees the algorithm does not make many query steps. At the beginning $\alpha(A_I) = 0$. In each communication step $\alpha(A_I)$ increases by at most 2 and in query step it decreases by at least $\log m/20$. Since the algorithm makes only $\mathsf{P}^{\mathsf{cc}}(F)$ communication step, it makes at most $40\mathsf{P}^{\mathsf{cc}}(F)/\log m$ query steps.

Lemma 5 (Thickness Lemma)**.** *If $n \geq 2$ and $A \subseteq [m]^n$ is such that $avg_i(A) \geq d$ for all $i \in [n]$, then there exists $A' \subseteq A$ such that:*

1. *For all i in $[n]$ it holds that $\delta_i(A') \geq \frac{d}{2n}$.*
2. *$\alpha(A') \leq \alpha(A) + 1$ ($\Leftrightarrow |A'| \geq \frac{1}{2}|A|$).*

Lemma 6 (Projection Lemma)**.** *Suppose $n \geq 2, A \subseteq [m[^n$ is thick and $B \subseteq (\{0,1\}^m)^n$ is such that $\beta(B) \leq m^{2/20}$. Then for every $i \in [n]$ and every $b \in \{0,1\}$ there exists a b-monochromatic rectangle $U \times V \subseteq [m] \times \{0,1\}^m$ in g such that:*

1. *$A^{i,U}_{[n]\setminus\{i\}}$ is thick.*
2. *$\alpha\big(A^{i,U}_{[n]\setminus\{i\}}\big) \leq \alpha(A) - \log m + \log avg_i(A)$.*
3. *$\beta\big(B^{i,V}_{[n]\setminus\{i\}}\big) \leq \beta(B) + 1$.*