Colouring quadrangulations of projective spaces

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The authors extend the definition of quadrangulation to higher dimensions and prove that every non-bipartite graph G which embeds as a quadrangulation in the n-dimensional real projective space P^n has chromatic number at least $n+2$.

Preliminaries:

- A graph that embeds in the real projective plane P^2 so that every face is bounded by a walk of length 4 is called a *projective quadrangulation.*
- In 1996, Youngs showed that the chromatic number of a projective quadrangulation is either 2 or 4.
- A topological space K (a subspace of some Euclidean space $\mathbb{R}^N$) is a *generalized simplicial complex* if K can be constructed using the following 'gluing process'. We start with discrete point space $K^{(0)}$ in $\mathbb{R}^N$ and at each step $i > 0$ we inductively construct the space $K^{(i)}$ by attaching a set of i-dimensional simplices to $K^{(i-1)}$. The images of the simplices involved in the construction are the *faces* of K. Each simplex is attached via a gluing map $f\colon \partial\Delta_i \to K^{(i-1)}$ that maps the interior of each face of the boundary of the standard i-simplex Δ_i in $\mathbb{R}^i$ homeomorphically to the interior of a face of $K^{(i-1)}$ of the same dimension. The *polyhedron* $\|K\|$ of K is the union of all faces of K.
- A *quadrangulation* of a generalized simplicial complex K is a spanning subgraph G of $K^{(1)}$ such that (inclusion-wise) maximal simplex of K induces a complete bipartite subgraph of G with at least one edge. If $\|K\|$ is homeomorphic to a topological space X, we say that the natural embedding of G in X is a *quadrangulation of* X.
- We say that K *triangulates* the space $\|K\|$ or any space homeomorphic to it.

Main results:

- The authors generalize the lower bound of Youngs.

  Theorem 1. *If G is a non-bipartite quadrangulation of P^n, then $\chi(G) \geq n+2$.*

- The authors show that the family of quadrangulations of projective spaces include all complete graphs and all (generalized) Mycielski graphs. In particular, the chromatic number of quadrangulations of P^n cannot be bounded from above for any $n > 2$.

  Theorem 2. *For $n \geq 3$ and $t \geq 5$, the complete graph K_t embeds in P^n as a quadrangulation if $t-n$ is even.*

- For positive integers n and k, the *Kneser graph* $KG(n,k)$ is a graph with the vertex set $\binom{[n]}{k}$ and with edges $\{A,B\}$ where $A,B \in \binom{[n]}{k}$ and $A \cap B = \emptyset$.
- We let $\binom{[n]}{k}_{stab}$ be the set of independent subsets of size k in the cycle C_n with the vertex set $[n]$. The *Schrijver graph* $SG(n,k)$ is a graph with the vertex set $\binom{[n]}{k}_{stab}$ and with edges $\{A,B\}$ where $A,B \in \binom{[n]}{k}_{stab}$ and $A \cap B = \emptyset$.

  Theorem 3. *Let $n > 2k$ and $k \geq 1$. There exists a non-bipartite quadrangulation of P^{n-2k} that is homomorphic to $SG(n,k)$.*

- Since $SG(n,k)$ is a subgraph of the Kneser graph $KG(n,k)$, Theorems 1 and 2 give an alternative proof of the *Lovász–Kneser theorem*, namely $\chi(KG(n,k)) \geq n-2k+2$.

Proof of Theorem 1:

- Let K be a generalized simplicial complex and p a non-negative integer. Restricting to $\mathbb{Z}_2$ coefficients, a *p-chain* of K is a finite formal sum of some of the p-simplices of K and the group of p-chains of K is denoted by $C_p(K, \mathbb{Z}_2)$. The *boundary* of a p-chain c is denoted by $\partial_p(c)$, where $\partial_p\colon C_p(K, \mathbb{Z}_2) \to C_{p-1}(K, \mathbb{Z}_2)$ is the *boundary operator*. The group of *p-cycles* of $C_p(K, \mathbb{Z}_2)$ is defined as $Z_p(K, \mathbb{Z}_2) := \mathrm{Ker}\partial_p$ and the group of *p-boundaries* of $C_p(K, \mathbb{Z}_2)$ as $B_p(K, \mathbb{Z}_2) := \mathrm{Im}\partial_{p+1}$. The *pth homology group* $H_p(K, \mathbb{Z}_2)$ is the quotient $Z_p(K, \mathbb{Z}_2)/B_p(K, \mathbb{Z}_2)$.

- Two p-cycles $c_1, c_2 \in Z_p(K, \mathbb{Z}_2)$ are *homologous* if there exists a $(p+1)$-chain d such that $c_1+c_2 = \partial_{p+1}(d)$.

  Lemma 4. *In every quadrangulation G of a topological space X, homologous cycles have the same parity; in particular, 0-homologous cycles are even. If $X = P^n$ and G is not bipartite, then every 1-homologous cycle is odd.*

- A generalized simplicial complex K is a *symmetric triangulation* of K if $-\sigma \in K$ for every face $\sigma \in K$.

- A *2-colouring* c of K is an arbitrary assignment of two colours to the vertices of K. We say that c is *proper* if there is no monochromatic maximal simplex. The graph *associated* to c is a spanning subgraph of $K^{(1)}$ consisting of all edges with vertices colored by distinct colors in c.

  Lemma 5. *For a graph G, the following statements are equivalent.*

  (a) The graph G is a non-bipartite quadrangulation of P^n.

  (b) There is a symmetric triangulation T of S^n such that no simplex of T contains antipodal vertices and there is a proper antisymmetric 2-colouring of T such that G is obtained from the associated graph by identifying all pairs of antipodal vertices.

- For a topological space X, a homeomorphism $\xi\colon X \to X$ is called a *$\mathbb{Z}_2$-action* on X if $\xi^2 = \xi \circ \xi = \mathrm{id}_X$. A $\mathbb{Z}_2$-action is *free* if it has no fixed points. A topological space X equipped with a (free) $\mathbb{Z}_2$-action is a *(free) $\mathbb{Z}_2$-space.*

- Given $\mathbb{Z}_2$-spaces (X, ξ) and (Y, ω), a continuous map $f\colon X \to Y$ such that $f \circ \xi = \omega \circ f$ is a *$\mathbb{Z}_2$-map.* If there exists a $\mathbb{Z}_2$-map from X to Y, we write $X \xrightarrow{\mathbb{Z}_2} Y$. The *$\mathbb{Z}_2$-coindex* of X is defined as

$$\mathrm{coind}(X) := \max\{n \geq 0\colon S^n \xrightarrow{\mathbb{Z}_2} X\}.$$

- Given a graph G, the set of *common neighbours* of a set $A \subseteq V(G)$ is defined as

$$\mathrm{CN}(A) := \{v \in V(G)\colon \{a, v\} \in E(G) \text{ for all } a \in A\}.$$

  The *box complex* of a graph G without isolated vertices is the simplicial complex with vertex set $V(G) \times \{1, 2\}$, defined as

$$B(G) := \{A_1 \uplus A_2\colon A_1, A_2 \subseteq V(G), A_1 \subseteq \mathrm{CN}(A_2) \neq \emptyset, A_2 \subseteq \mathrm{CN}(A_1) \neq \emptyset\},$$

  where we use the notation $A \uplus B$ for the set $(A \times \{1\}) \cup (B \times \{2\})$.

  The box complex is equipped with a natural free $\mathbb{Z}_2$-action ω which interchanges the two copies of $V(G)$, namely $\omega\colon (v, 1) \mapsto (v, 2)$ and $\omega\colon (v, 2) \mapsto (v, 1)$.

  Theorem 6 (Lovász, 1978)**.** *If G is a graph with no isolated vertices, then $\chi(G) \geq \mathrm{coind}(B(G)) + 2$.*