Threesomes, Degenerates, and Love Triangle

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In 3Sum problem, given n real numbers one have to decide whether any three numbers sum to zero. The 3SUM conjecture gives a $\Omega(n^2)$ lower bound. In this paper 3SUM conjecture is refuted by showing subquadratic decision tree and algorithmic complexity for 3SUM. This result also gives improved bounds for k linear degeneracy testing for all odd k $\geq$ 3. A subcubic algorithm for generalized (min,+) product has also been proven which indeed helps to find zero weight triangles in weighted graphs.

Problems :

- **3 SUM :** Given a set $A \subset \mathbb{R}$, determine if there exist a, b, c $\in$ A such that a+b+c=0.
- **Integer 3SUM :** Given a set $A \subseteq \{-U, ..., U\} \subset \mathbb{Z}$, determine if there exist a, b, c $\in$ A such that a+b+c=0.
- **k-LDT and k-SUM :** Fix a k-variate linear function $\phi(x_1, ..., x_k) = \alpha_0 + \Sigma_{i=1}^{1} \alpha_i x_i$, where $\alpha_0, ..., \alpha_k \in \mathbb{R}$. Give a set $A \subset \mathbb{R}$, determine is $\phi(x) = 0$ for any $x \in A^k$. when ϕ is $\Sigma_{i=1}^{1} x_i$ the problem is called k-SUM.
- **Zero Triangle :** Given a weighted undirected graph G=(V, E, w), where $w : E \to \mathbb{R}$, determine if there exist a triangle $(a, b, c) \in V^3$ for which w(a,b)+w(b,c)+w(c,a)=0.
- **Convolution3SUM :** Given a vector $A \in \mathbb{R}^n$, determine if there exist i.j for which $A(i) + A(j) = A(i + j)$.
- **IntegerConv3SUM :** The same as Convolution3SUM, except that $A \in \{0, ..., U - 1\}^n$ and $U \leq 2^w$, where $w = \Omega(logn)$ is the machine word size.
- **(min, +)-product :** Given real matrices $A \in (\mathbb{R} \cup \{\infty\})^{r \times s}, B \in (\mathbb{R} \cup \{\infty\})^{s \times t}$, and a target matrix $T \in (\mathbb{R} \cup \{\infty\})^{r \times t}$, the goal is to compute $C = \circledcirc(A, B, T)$, where

$$C(i, j) = min\{A(i, k) + B(k, j) | k \in [s] and A(i, k) + B(k, j) \geq T(i, j)\}$$

Some Reductions :

- Patrascu defined Convolution3SUM and gave reduction from Integer3SUM to Convolution3SUM.
- William gave a reduction from Convolution3SUM to Zero Triangle.
- Hence an $O(n^2)$ bound for zero triangle gives $O(n^{9/5})$ bound on Integer3SUM.

Lemmas Used :

Lemma 1.(*Fredman 1976*) A list of n numbers whose sorted order is one of $\prod$ permutations can be sorted with $2n + log \prod$ pairwise comparisons.

Lemma 2.(*Buck 1943*) Consider the partition of space defined by an arrangement of m hyperplanes in $\mathbb{R}^d$. The number of regions of dimension $k \leq d$ is at most

$$\binom{m}{d-k} \quad \left(\binom{m-d+k}{0} \quad + \quad \binom{m-d+k}{1} \quad + ... + \quad \binom{m-d+k}{k} \right)$$

and the number of regions of all dimensions is $O(m^d)$.

Lemma 3. Let $A = a_{ii\in[n]}$ be two lists of numbers and let $F \subseteq [n]^2$ be a set of positions in $n \prod n$ grid. the number of realizable orders of $(A + B)_{|F} = \{a_i + b_j | (i, j) \in F\}$ is O($\binom{|F|}{2}^{2n}$) and therefore $(A+B)_{|F}$ can be sorted with at most $2|F|+2nlog|F|+O(1)$ comparisons.

Lemma 4. Let $A = (a_i)$ and $B = (b_i)$ be two list of numbers. Define $a_i^{'} = (a_i, i, 0)$ and $b_j^{'} = (b_j, 0, j)$. The Cartesian sum A'+B' is totally ordered, and is a linear extension of the partially ordered A+B. (Addition over tuples is pointwise addition; tuples are ordered lexicogaphically. The tuple (u, v, w) can be regarded as a representation of a real number $u + \epsilon_1 v + \epsilon_2 w$ where $\epsilon_1 \gg \epsilon_2$ are sufficiently small so as not to invert strictly ordered elements A+B)

Lemma 5. (Biochromatic Dominance Reporting) Given a set $P \subseteq \mathbb{R}^d$ of red and blue points, it is possible to return all biochromatic dominating pairs $(p, q) \in P^2$ in time linear in the output size and $c_\epsilon^d |P|^{1+\epsilon}$. Here $\epsilon \in (0, 1)$ is arbitrary and $c_\epsilon = 2^\epsilon/(2^\epsilon - 1)$.

Results :

Theorem1. There is a 4-linear decision tree for 3SUM with depth $O(n^{3/2}\sqrt{\text{log n}})$. Furthermore, 3SUM can be solved deterministically in $O(n^2/(\text{log n}/\text{log log n})^{2/3})$ time and in $O(n^2(\text{log log n})^2/\text{log n})$ time using randomization.

Theorem2. When $k \geq 3$ is odd, there is a (2k-2)-linear decision tree for k-LDT with depth $O(n^{k/2}\sqrt{\text{log n}})$.

Theorem3. The decision tree complexity of Zero Triangle is $O(n^{5/2}\sqrt{\text{log n}})$ on n-vertex graph and randomized decision tree complexity is $O(n^{5/2})$ with high probability. Also it can be solved deterministically in $O(n^3(\text{log log n})^2/\text{log n})$ time and in time $O(n^3 \text{ log log n}/\text{log n})$ time using randomization with high probability.

Theorem4. The decision tree complexity of Zero Triangle on m-edge graphs is $O(m^{5/4}\sqrt{\text{log m}})$ and, using randomization, $O(m^{5/4})$ with high probability. It can be solved in $O(m^{3/2}(\text{log log m})^2/\text{log m})$ time deterministically or $O(m^{3/2}\text{log log m}/\text{log m})$ with high probability.

Theorem5. The decision tree complexity of Convolution3SUM is $O(n^{3/2}\sqrt{\text{log n}})$ and its randomized decision tree complexity is $O(n^{3/2})$ with high probability. The Convolution3SUM problem can be solved in $O(n^2(\text{log log n})^2/\text{log n})$ time deterministically, or in $O(n^2 \text{ log log n}/\text{log n})$ time with high probability.