The matching polytope has exponential extension complexity

Thomas Rothvoß

November 27, 2014

presented by Vojtěch Kaluža

Notation.

- *The formulas $F \sim \mathcal{A}$ and $F \in_{\mathcal{U}} \mathcal{A}$ means that F is a uniformly random element of $\mathcal{A}$.*
- *We use only uniform distribution, no other probability distributions are considered.*

Definition (Perfect matching polytope): $P_M = \mathrm{conv}\{\chi_M \in \mathbb{R}^E \mid M \subseteq E \text{ is a perfect matching}\}$ $\overset{Edmonds}{=} \{x \in \mathbb{R}^E \mid x(\delta(v)) = 1\ \forall v \in V; x(\delta(U)) \geq 1\ \forall U \subseteq V, |U| \text{ odd}\ ; x_e \geq 0\ \forall e \in E\}$.

Definition (Extension complexity): *The* ***extension complexity*** $\mathrm{xc}(P)$ *of a polytope P is defined as the minimal number of facets of a higher dimensional polytope Q s.t. there is a linear projection π satisfying $\pi(Q) = P$.*

Theorem 1 (Rothvoß): *For all $n \in \mathbb{N}$, $\mathrm{xc}(P_M) \geq 2^{\Omega(n)}$ in the complete n-node graph.*

Fact: *If P is a linear projection of a face of P', then $\mathrm{xc}(P) \leq \mathrm{xc}(P')$.*

Corollary 2: *Because Yannakakis described a linear projection of a face of P_{TSP} of $O(n)$-node complete graph onto P_M of n-node complete graph, we have $\mathrm{xc}(P_{TSP}) \geq 2^{\Omega(n)}$, too.*

Definition (Slack matrix): *Let $P = \mathrm{conv}\{x_1, \dots, x_v\} = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, where $A \in \mathbb{R}^{f \times n}$. We define the* ***slack matrix*** *$S \in \mathbb{R}^{f \times v}_{\geq 0}$ as $S_{ij} = b_i - A_i x_j$, i.e. S_{ij} is the slack of the j-th vertex in the i-th inequality.*

Definition (Non-negative rank): *The* ***non-negative rank*** $\mathrm{rk}_+(S)$ *of a matrix S is defined as* $\mathrm{rk}_+(S) = \min\{r \mid (\exists U \in \mathbb{R}^{f \times r}_{\geq 0})(\exists V \in \mathbb{R}^{r \times v}_{\geq 0})(S = UV)\}$.

Theorem 3 (Yannakakis 91): *Let P be a polytope with vertices $\{x_1, \dots, x_v\}$, $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$, let S be its slack matrix. Then*

- $\mathrm{xc}(P) = \mathrm{rk}_+(S)$
- *Moreover, the minimal extended formulation of P can be obtained by factoring $S = UV$, where U, V come from the definition of* rk_+*, and writing* $P = \{x \in \mathbb{R}^n \mid (\exists y \geq 0)(Ax + Uy = b)\}$

Lemma 4 (Hyperplane separation lower bound): *Let $S \in \mathbb{R}^{f \times v}_{\geq 0}$ be the slack matrix of a polytope P and $W \in \mathbb{R}^{f \times v}$ be any matrix. Then*

$$\mathrm{xc}(P) \geq \frac{\langle W, S \rangle}{\|S\|_\infty \cdot \alpha} \qquad \text{where } \alpha = \max\{\langle W, R \rangle \mid R \in \{0,1\}^{f \times v} \text{ rank 1-matrix}\}.$$

Notation.

- *Choose a parameter $k > 3$ odd, $t = \frac{m+1}{2}(k-3)+3$ is an odd magic constant.*
- $\mathcal{M}_{all} = \{M \subseteq E \mid M \text{ is a perfect matching}\}$
- $\mathcal{U}_{all} = \{U \subseteq V \mid |U| = t\}$
- $Q_l = \{(U, M) \in \mathcal{M}_{all} \times \mathcal{U}_{all} \mid |\delta(U) \cap M| = l\}$, *$\mu_l$ the uniform measure on Q_l*
- *Rectangle $\mathcal{R} = \mathcal{U} \times \mathcal{M}$ where $\mathcal{U} \subseteq \mathcal{U}_{all}$ and $\mathcal{M} \subseteq \mathcal{M}_{all}$*

Lemma 6: *For all $k > 3$ odd and for all rectangles $\mathcal{R}$ with $\mu_1(\mathcal{R}) = 0$ we have that $\mu_3(\mathcal{R}) \leq \frac{400}{k^2} \cdot \mu_k(\mathcal{R}) + 2^{-\delta m}$, where $\delta = \delta(k) > 0$ is a constant.*

Definition (Partition): *A partition is a tuple $T = (A = \dot\bigcup_{i=1}^m A_i, C, D, B = \dot\bigcup_{i=1}^m B_i)$ with $V = A \,\dot\cup\, C \,\dot\cup\, D \,\dot\cup\, B$, $|C| = |D| = k$ and $|A_i| = k-3$, $|B_i| = 2(k-3)$ for every $i \in [m]$.*

Notation.

- $E(T) = \dot\bigcup_{i=1}^m E(A_i) \,\dot\cup\, E(C \cup D) \,\dot\cup\, \dot\bigcup_{i=1}^m E(B_i)$
- $\mathcal{M}(T) = \{M \in \mathcal{M} \mid M \subseteq E(T)\}$
- $\mathcal{M}_{all}(T) = \{M \in \mathcal{M}_{all} \mid M \subseteq E(T)\}$
- $\mathcal{U}(T) = \{U \in \mathcal{U} \mid U \subseteq A \cup C \text{ with } |U \cap A_i| \in \{0, |A_i|\}\ \forall i \in [m]\}$
- $\mathcal{U}_{all}(T) = \{U \in \mathcal{U}_{all} \mid U \subseteq A \cup C \text{ with } |U \cap A_i| \in \{0, |A_i|\}\ \forall i \in [m]\}$
- $p_{\mathcal{M},T}(H) = \Pr_{M \sim \mathcal{M}_{all}(T)}[M \in \mathcal{M} \mid H \subseteq M]$
- $p^{ex}_{\mathcal{M},T}(H) = \Pr_{M \sim \mathcal{M}_{all}(T)}[M \in \mathcal{M} \mid M \cap \delta(C) = H]$
- $p_{\mathcal{U},T}(c) = \Pr_{U \sim \mathcal{U}_{all}(T)}[U \in \mathcal{U} \mid c \subseteq U]$, *where $c \subseteq C$*
- $p^{ex}_{\mathcal{U},T}(c) = \Pr_{U \sim \mathcal{U}_{all}(T)}[U \in \mathcal{U} \mid C \cap U = c]$, *where $c \subseteq C$*
- $p^{ex}_{\mathcal{U},T}(H) = p^{ex}_{\mathcal{U},T}(V(H) \cap C)$ *for a matching $H \subseteq C \times D$*

Definition ($\mathcal{M}$-good): *Let T be a partition and H a 3-matching in $C \times D$. Then (T, H) is called* ***$\mathcal{M}$-good*** $\iff 0 < \frac{1}{1+\epsilon} p_{\mathcal{M},T}(H) \leq p_{\mathcal{M},T}(F) \leq (1+\epsilon) p_{\mathcal{M},T}(H)$ *for every k-matching $F \subseteq E(C \cup D)$ s.t. $H \subseteq F$. Otherwise, it is called* ***$\mathcal{M}$-bad****.*

Definition ($\mathcal{U}$-good): *Let T be a partition and H a 3-matching in $C \times D$. Then (T, H) is called* ***$\mathcal{U}$-good*** $\iff 0 < \frac{1}{1+\epsilon} p^{ex}_{\mathcal{U},T}(H) \leq p^{ex}_{\mathcal{U},T}(C) \leq (1+\epsilon) p^{ex}_{\mathcal{U},T}(H)$. *Otherwise, it is called* ***$\mathcal{U}$-bad****.*

Definition (Good): *If (T, H) is both $\mathcal{U}$-good and $\mathcal{M}$-good, we call it just* ***good****.*

Lemma 7: *If (T, H) is $\mathcal{M}$-good $\implies \frac{1}{1+\epsilon} p_{\mathcal{M},T}(H) \leq p^{ex}_{\mathcal{M},T}(H) \leq (1+\epsilon) p_{\mathcal{M},T}(H)$.*

Lemma 8: *If T is a partition and $F \subseteq C \times D$ is a k-matching, then $\Pr_{H \sim \binom{F}{3}}[(T, H) \text{ is good}] \leq \frac{100}{k^2}$*

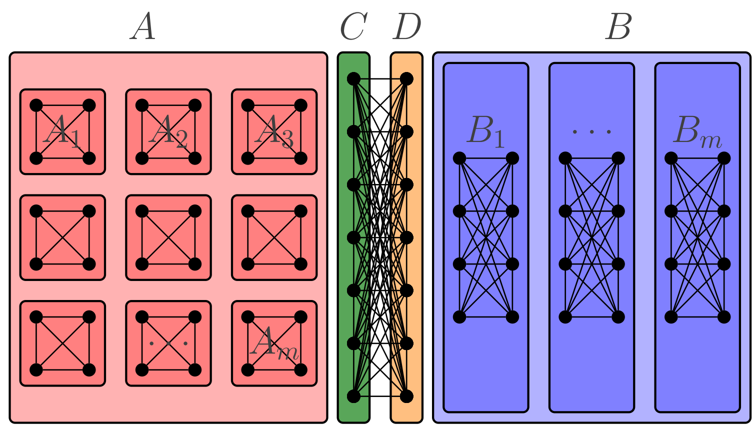

Figure 1: Visualization of a partition T with all edges $E(T)$.

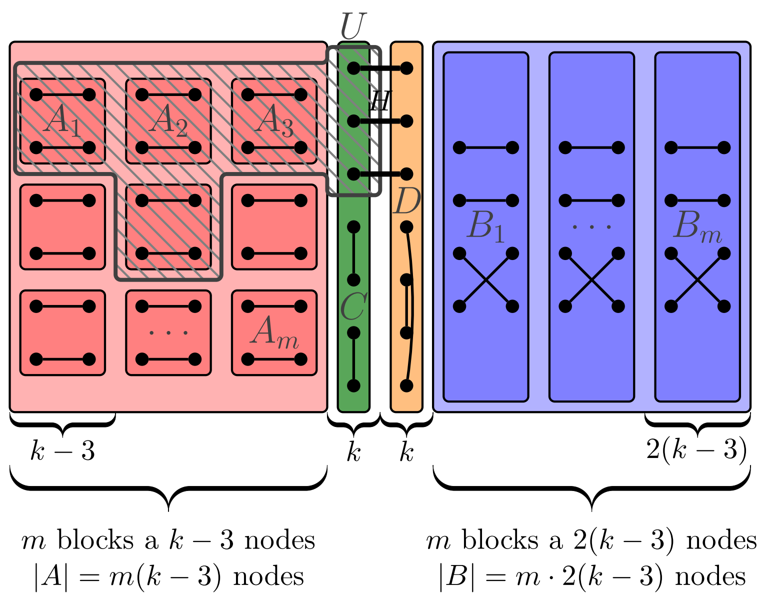

Figure 2: Visualization of a partition T together with one matching $M \in \mathcal{M}(T)$ and one cut $U \in \mathcal{U}(T)$.