The Cover Number of a Matrix and its Algorithmic Applications

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Let $A \in [-1;1]^{m \times n}$ be a matrix. We consider a quadratic optimization problem where we maximize $p^T Aq$ over probability distributions p and q subject to linear constraints.

Basic definitions:

- $\Delta^n = \{p \in [0;1]^n : \|p\|_1 = \sum_{i=1}^n p_i = 1\}$ is the set of n-dimensional probability distributions,
- $\mathrm{conv}(A)$ is the convex hull of the columns of A,
- ε-net for A is the set of vectors $S \subseteq \mathbb{R}^m$ such that for all $v \in \mathrm{conv}(A)$ there is a vector $u \in S$ satisfying $\|v - u\|_\infty \leq \varepsilon$,
- The cover number $N_\varepsilon(A)$ is the minimal size of an ε-net for A.

Approximation framework: Given an efficient enumerator for an ε-net S solve for each $u \in S$ the linear program max $p^T u$ over $p \in \Delta_m, q \in \Delta_n$ subject to original linear constraints and $\|u - Aq\|_\infty \leq \varepsilon$. This yields a solution which is within 2ε of the optimal.

Application: Approximate Nash equilibria

In a 2-player game let $A, B \in [-1;1]^{m \times n}$ be payoff matrices for Alice and Bob respectively, i.e., $A_{i,j}$ is payoff for Alice when she plays strategy i and Bob plays strategy j. Let $p \in \Delta_m, q \in \Delta_n$ be mixed strategies for Alice and Bob respectively. The pair of strategies p, q is a Nash equilibrium (NE) if it satisfies

$$p^T Aq \geq e_i^T Aq \quad \forall i \in [m] = \{1, \ldots, m\}$$
$$p^T Aq \geq p^T Ae_j \quad \forall j \in [n] = \{1, \ldots, n\},$$

i.e., neither Alice, nor Bob can improve his or her payoff by changing the mixed strategy to a different pure strategy (assuming that the other one stick to his or her strategy). The ε-Nash equilibrium is similar to NE, but they can improve by at most ε.

Theorem 1. *Using a deterministic (or Las Vegas randomized) algorithm for enumerating $\varepsilon/2$-net for $A+B$ (running in time t) we can find an ε-Nash equilibrium in time $t \cdot \mathrm{poly}(mn)$.*

Upper bounds on the cover number

Quasi-polynomial upper bound

Theorem 2. *Let $A \in [-1;1]^{m \times n}$ be a matrix. Then $N_\varepsilon(A) \leq \binom{n+k}{k} < n^k$ where $k = 2\ln(2m)/\varepsilon^2$.*

Upper bound using VC dimension

Definition. Let $A \in \mathbb{R}^{m \times n}$ be a matrix. Let $C = \{c_1, \ldots, c_k\} \subseteq [n]$ be a subset of columns of A. We say that A *shatters* C if there are real numbers $(t_{c_1}, \ldots, t_{c_k})$ such that for any $D \subseteq C$ there is a row i with $A_{i,c} < t_c$ for all $c \in D$ and $A_{i,c} > t_c$ for all $c \in C \setminus D$.

Let $\mathrm{VC}(A)$ be the maximal size of a set of columns shattered by A (Vapnik–Chervonenkis dimension or pseudo-dimension).

Theorem 3. *Let $A \in [-1;1]^{m \times n}$ be a matrix with $\mathrm{VC}(A) = d$. Then*

$$N_\varepsilon(A) \leq n^{\mathcal{O}(d/\varepsilon^2)}.$$

Lower bounds on the cover number

Lemma 4. *Let $A \in \{-1;1\}^{m \times n}$ be a sign matrix and $\mathcal{F}$ be a family of subsets of $[n]$ such that for every distinct $F, F' \in \mathcal{F}$*

1. *the columns of A in $F \cup F'$ are shattered,*
2. *$|F \cap F'| \leq (1-\delta)|F|$.*

Then $N_\delta(A) \geq |\mathcal{F}|$.

Theorem 5. *Let $A \in \{-1;1\}^{m \times n}$ be a sign matrix. Then $N_{1/4}(A) \geq 2^{\Omega(\mathrm{VC}(A))}$.*

Theorem 6. *For almost all sign matrices $A \in \{-1;1\}^{n \times n}$ it holds that $N_{0.99}(A) \geq n^{\Omega(\log n)}$.*