Number of cliques in graphs with forbidden subdivison

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Well-known facts

Definition 1. *Graph G is k-degenerated if every induced subgraph $G' \subseteq G$ contains vertex of degree at most k.*

Theorem 2. *Let G be K_t-minor-free graph. Then G has average degree at most $ct\sqrt{\ln t}$.*

Theorem 3. *Let G be K_t-subdivision-free graph. Then G has average degree at most $10t^2$.*

If graph is H-minor-free (resp. H-subdivision-free) then all of its subgraphs are. Therefore K_t-minor-free graphs are $(ct\sqrt{\ln t})$-degenerated and K_t-subdivision-free graphs are $(10t^2)$-degenerated.

Let G be n-vertex d-degenerate graph ($n \geq d$). Then G has at most $2^d(n-d+1)$ cliques.

Theorem 4. *Let G be K_t-minor-free (resp. K_t-subdivision-free) graph. Then G has at most $2^{ct\sqrt{\ln t}}n$ (resp. $2^{10t^2}n$) cliques.*

Improving upper bound

Theorem 5. *Let G be n-vertex graph without K_t-subdivison. Then G has at most $2^{94t}n$ ($2^{5t+o(t)}n$ asymptotically) cliques.*

Function `buildCST(`G`)`
    Input: Graph G
    Output: Clique search tree of G
    Create tree T with single node r as root. Set label $L(r) \leftarrow V(G)$.
    while $|V(G)| > 0$ **do**
        $v \in V(G)$ s.t. v has minimum degree
        Add `buildCST(`$G[N(v)]$`)` as child of r and set v as label for the new edge.
        $G \leftarrow G - v$
    end while
    return T
end

Algorithm 1: Creation of Clique search tree.

Observation 6. *Let T be clique search tree of G. Then each node of T corresponds to clique in G consisting of label of edges on path from root to given node.*

Definition 7. *Rooted subtree is subtree containing root. Boundary of rooted subtree T' are all nodes of T' that are adjacent to node not in T'.*

Definition 8. *Graph G is (β, N)-locally sparse if every set X of at least N vertices contains vertex of degree at most $\beta\,|X|$ in $G[X]$.*[1]

Lemma 9. *Let G be n-vertex K_t-subdivision-free graph. If G has minimum degree at least $\frac{9}{10}n$, then $n \leq \max\left\{\frac{20}{11}t, \frac{t^2}{5}\right\}$ and G is $(1 - \frac{n}{2t^2}, \frac{20}{11}t)$-locally sparse.*

Lemma 10. *Let G be n-vertex K_t-subdivision-free graph with $n \leq \frac{t^2}{5}$. If G is $(1-\frac{n}{2t^2}, \frac{20}{11}t)$-locally sparse, then G contains less than $t2^{5t}$ cliques.*

[1] Slightly weaker than usual definition.