Conjectures

Conj (Dean)**.** *In every tournament there exists a vertex* v*, such that* $|N^{++}(v)| \geq |N^{+}(v)|$*.*

Conj (Seymour)**.** *In every* **digraph** *there exists a vertex* v*, such that* $|N^{++}(v)| \geq |N^{+}(v)|$*.*

Conj (Summers)**.** *For* $n > 1$*, every tournament of order* $2n-2$ *contains every oriented tree of order* n*.*

Definitions

Definition 1. Let $T = (V, E)$ be a tournament and $L = (V, E')$ be a total order on V. Denote by $T \cap L$ the acyclic directed graph $(V, E \cap E')$. An order L of T which maximizes the number of arcs of $T \cap L$ is a *median order of* T.

Feedback Property for every i, j with $1 \leq i \leq j \leq n$: outdegree of x_i and indegree of x_j in $(T \cap L)_{|[x_i, x_j]}$ are at least $(j-i)/2$

Definition 2. A *local median order* of T is an order of T which satisfies the feedback property.

Inductive tool if I is an interval of a (local) median order L of T, then $L_{|I}$ is a (local) median order of $T_{|I}$.
Vertex v of a tournament T is

feed vertex if there exists a local median order L of T such that v is maximal in L

back vertex if there exists a local median order L of T such that v is minimal in L

dominating if $d^{-}(v) = 0$

dominated if $d^{+}(v) = 0$

king if $\{v\} \cup N^{+}(v) \cup N^{++}(v) = V(T)$

Definition 3. Let $L = (x_1, \ldots, x_n)$ be a local median order of a tournament T. We distinguish two types of vertices of $N^{-}(x_n)$: a vertex $x_j \in N^{-}(x_n)$ is *good* if there exists $x_i \in N^{+}(x_n)$, with $i < j$ such that $x_i \to x_j$; otherwise x_j is *bad*. We denote the set of good vertices of (T, L) by G_L.

Definition 4. Let $L = (x_1, \ldots, x_n)$ be a local median order of a tournament T. A *sedimentation* of a median order L is denoted by $Sed(L)$.

- If $|N^+(x_n)| < |G_L|$, then $Sed(L) = L$.
- If $|N^+(x_n)| = |G_L|$, we denote by $b_1, \ldots, b_k$ the bad vertices of (T, L) and by $v_1, \ldots, v_{n-1-k}$ the vertices of $N^+(x_n) \cup G_L$, both enumerated in increasing order with respect to their index in L. In this case, $Sed(L) = (b_1, \ldots, b_k, x_n, v_1, \ldots, v_{n-1-k})$.

Definition 5. A rooted tree with all edges oriented towards the root is called *arborescence*.

Definition 6. An *embedding of an arborescence A into a tournament T* is an injective mapping $f : V(A) \to V(T)$ such that $f(x) \to f(y)$ whenever $x \to y$.

Definition 7. A directed graph D is *m-unavoidable* if for every tournament T of order m, there exists an embedding of D into T.

Definition 8. An embedding of A into T is an L-up-embedding if $|N_L^+(x) \cap f(A)| \leq |N_L^+(x) \setminus f(A)| + 1$.

Definition 9. A tree A is *m-well-up-embeddable* if for every tournament of order m and every local median order L A is L-up-embeddable.

Warm up

Proposition 1. *Every tournament has a king. Moreover, a tournament with no dominating vertex has at least three kings.*

Theorem 1. *Every feed vertex of a tournament has a large second neighbourhood.*

First Theorems

Lemma 1. *The order $Sed(L)$ is a median order of T.*

Theorem 2. *A tournament with no dominated vertex has at lest two vertices with large second neighbourhood.*

And Beyond

Theorem 3. *Every arborescence of order $n > 1$ in $(2n-2)$-unavoidable.*

Theorem 4. *Every tree of order $n > 1$ is $(4n-6)$-unavoidable.*

Theorem 5. *Every tree of order $n > 0$ is $(\frac{7n-5}{2})$-unavoidable.*