How Many n-vertex Triangulations Does the 3-sphere Have?

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Known upper $2^{O(n^{\lceil \frac{d}{2} \rceil} \log(n))}$ and lower $2^{\Omega(n^{\lfloor \frac{d}{2} \rfloor})}$ bound for the number of combinatorially distinct n-vertex triangulations of d-spheres. For 3-spheres gap very big, we show how to construct at least $2^{\Omega(n^2)}$ such triangulations.

Theorem. 1.1 *For each $n \geq 1$ there exists a 3-dimensional polyhedral sphere with $5n+4$ vertices, such that n^2 of its facets are combinatorially equivalent to a bipyramid.*

Corollary. 1.2 The 3-sphere admits $2^{\Omega(n^2)}$ combinatorially distinct triangulations on n vertices.

A Few Terms from Algebraic Topology

- **simplex** $C = \{\theta_0 u_0 + \cdots + \theta_k u_k | \theta_i \geq 0, 0 \leq i \leq k, \sum_{i=0}^{k} \theta_i = 1\}$ – generalization of triangle and tetrahedron to arbitrary dimensions
- **simplicial complex** $\mathcal{X}$ is a set of simplexes that satisfies
  - any face of a simplex from $\mathcal{X}$ is also in $\mathcal{X}$.
  - the intersection of any two simplexes $\sigma_1, \sigma_2 \in \mathcal{X}$ is a face of both σ_1 and σ_2.
- **face** is the convex hull of any nonempty subset of the n+1 points that define an n-simplex
- $\mathcal{X}$ is said to be **pure** if all its maximal (w.r.t. inclusion) faces have the same dimension
- **facet** of a simplex is a face of maximal dimension (for a simplex $n-1$-faces)
- **facet** of a simplicial complex is any simplex, which is not par of any larger simplex
- k**-complex** if all facets are k**-faces** i.e. faces of dimension k
- $\partial\mathcal{X}$ – the **boundary** complex of $\mathcal{X}$ is a subcomplex of $\mathcal{X}$ which contains those faces which are in ! one facet of $\mathcal{X}$
- a face F is **interior** if $F \notin \partial\mathcal{X}$
- **link** of a face F is a subcomplex $\{T \in \mathcal{X}; T \cap F = \emptyset; T \cup F \in \mathcal{X}\}$
- **star** of a face F is a subcomplex $\{T \in \mathcal{X}; F \subseteq T; T facet \in \mathcal{X}\}$

- two complexes are said to be **combinatorially equivalent** / **isomorphic** if their lattices (partial order by set containment of faces) are isomorphic.
- n**-sphere** $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = r\}$
- n**-ball** $B^n = \{x \in \mathbb{R}^{n+1} : \|x\| \leq r\}$
- $(n-1)$-sphere $= \partial$ n-ball
- we will consider complexes which are only homeomorphic to spheres & balls

- **moment curve** in $\mathbb{R}^d$ is a curve $\alpha_d : \mathbb{R} \to \mathbb{R}^d$ defined $\alpha_d(t); = (t, t^2, t^3, \ldots, t^d)$
- **cyclic** d**-polytope** $C(n,d)$ is the convex hull of the n points $\alpha_d(1), \ldots, \alpha_d(n)$

Lemma. Gail's evenness condition All facets of $C(n,d)$ are $(d-1)$-simplices. Furthermore, for any set of d integers $I \subset [n]$, the convex hull $conv(\alpha_d(I))$ is a facet of $C(n,d)$ iff for every $x, y \in [n] \setminus I$, there are an even number of elements $z \in I$ satisfying $x < z < y$.

Construction of the Polyhedral Sphere

- we take cyclic 4-polytope $C(4n+4,4)$
- $P(n)$ polyhedral complex comb. isomorphic to $\partial C(4n+4,4)$
- $P(n)$ is homeomorphic to a 3-sphere
- $A(n) = \{m \in [n+2, 3n+1]; m = 2k, k \in \mathbb{Z}\}$
- facets of $P(n)$
  - $I(a,u,1) := \{a-u-1, a-u, a+u, a+u+1\}$
  - $I(a,u,2) := \{a-u-1, a-u, a+u+1, a+u+2\}$
  - $I(a,u,3) := \{a-u, a-u+1, a+u+1, a+u+2\}$

Lemma. For all $a \in A(n), u, u' \in [n], i, j \in [3]$, if $u' \leq u-1$ then

$$I(a,u,i) \cap I(a,u',j) \subseteq \begin{cases} \{a-u, a+u, a+u+1\}, & i=1 \\ \{a-u, a+u+1\}, & i=2 \\ \{a-u, a-u-1, a+u+1\}, & i=3 \end{cases}$$

- $B_0(a) := \{I(a,u,1); u \in [n], i \in [3]\}$, $B(a)$ is the closure of $B_0(a)$ under subsets
- **shelling** is an ordering $F_1, F_2, \ldots F_p$ of the maximal simplexes of $\mathcal{X}$ such that the complex $\mathcal{B}_k := \left(\bigcup_{i=1}^{k-1} F_i\right) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k = 2, 3, \ldots p$.

Lemma. 3.2 For each $a \in A(n)$, the simplicial complex $B(a)$ is a shellable simplicial 3-ball.

Lemma. 3.3 For distinct each $a, a' \in A(n)$, the intersection $B(a) \cap B(a')$ does not contain a 2-face of $P(n)$.

Lemma. 3.4 For distinct $a, a' \in A(n)$, we have $B(a) \cap B(a') \subset \partial B(a) \cap \partial B(a')$.

- some new notation:
  $x_-(a,u,1) := a-u-1 \quad x_+(a,u,1) := a+u$
  $x_-(a,u,2) := a-u-1 \quad x_+(a,u,2) := a+u+2$
  $x_-(a,u,3) := a-u+1 \quad x_+(a,u,3) := a+u+2$

Lemma. 3.5 For every $a \in A(n)$, the 2-faces of the boundary complex $\partial B(a)$ are exactly the triangles $I_\sigma(a,u,i) \cup \{x_{-\sigma(a,u,i)}\}$, for $u \in [n], i \in [3], \sigma \in \{+,-\}$.

- $E(a,u) := \{a-u, a+u+1\}$

Lemma. 3.6 The interior edges of $B(a)$ are exactly the edges $\{E(a,u) : u \in [n]\}$.

- $T_\sigma(a,u) := I_\sigma(a,u,1) \cup \{x_{-\sigma}(a,u,1)\}$ triangles without $E(a,u) = \{a-u, a+u+1\}$
- $D(a,u) :=$ closure of $\{T_-(a,u), T_+(a,u)\}$ under subsets (a 2-ball)
- $R(a,u) := T_-(a,u) \cap T_+(a,u) = \{a-u-1, a+u\}$ (unique interior edge in $D(a,u)$)

Lemma. 3.7 For $(a,u) \neq (a',u')$ the disks $D(a,u)$ and $D(a',u')$ intersect in a single face. When $a = a'$, this intersection lies on the boundary of both disks.