Handouts: Analytical Approach to Parallel Repetition

Elazar Goldenberg

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1 Notations and definitions

Definition 1.1. *1. A one round two prover game G game is specified by a bipartite graph with vertex sets U and V such that each edge (u, v) is endowed with a constraint $\pi_{(u,v)} \subset \Sigma \times \Sigma$.*

$$\mathsf{val}(G) := \max_{f,g} \mathsf{val}(G, f, g) := \max_{f,g} = \Pr_{(u,v)}[(f(u), g(v)) \in \pi_{u,v}].$$

2. *A game is called projection game if for every edge (u, v) and for every possible answer of Bob $\beta \in \Sigma$, there exists a single answer $\alpha \in \Sigma$ for $\mathcal{A}$ such that the constraint $\pi_{(u,v)}$ is satisfied.*

3. *In the k-folded parallel repetition of the game $G^{\otimes k}$ the referee chooses k edges $(u_1, v_1), \cdots (u_k, v_k)$ independently from G, and sends $(u_1, \cdots, u_k)$ to $\mathcal{A}$, and $(v_1, \cdots, v_k)$ to $\mathcal{B}$. Each of the players answers with a k-tuple string and they win if their answers satisfies each of the constraints on these edges.*

Theorem 1.2 (Parallel Repetition Theorem). *For any projection game G, if $\mathsf{val}(G) = 1 - \epsilon$ then $\mathsf{val}(G^{\otimes k}) \leq (1 - c\epsilon^2)^{k/2}$.*

2 Linear Algebra Notations:

Denote by $L(U \times \Sigma)$ the set of real-valued functions defined on $U \times \Sigma$. An assignment for $\mathcal{A}$ is specified by a function $g \in L(U, \Sigma)$ satisfying (i) $g(u, \alpha) \geq 0$, (ii) for every $u \in U$: $\sum_{\alpha,u} g(u, \alpha) = 1$ (Note that $\mathcal{A}$ maybe non-deterministic, on a query u she outputs α with probability $\frac{g(u,\alpha)}{\sum_{\alpha'} g(u,\alpha')}$).

For every projection game G we define a linear operator from $L(V \times \Sigma)$ to $L(U \times \Sigma)$ by:

$$Gf(u, \alpha) = \mathbb{E}_{v|u} \sum_{\beta:\alpha\leftarrow\beta} f(v, \beta), \text{ where } \alpha \leftarrow \beta \text{ means that } (\alpha, \beta) \in \pi(u, v).$$

We define an inner product $\langle g, g' \rangle = \mathbb{E}_u \sum_\alpha g(u, \alpha) g'(u, \alpha)$.

Claim 2.1. *1. $Gf(u, \alpha) \geq 0$, $\sum_\alpha f(u, \alpha) \leq 1$.*

2. *$\mathsf{val}(G, f, g) = \langle g, Gf \rangle$.*

3. *For any two projections games G, H, let $f \in L(U \times \Sigma \times U' \times \Sigma')$ we define:*

$$G \otimes Hf(u, \alpha, u', \alpha') = \mathbb{E}_{v|u,v'|u'} \sum_{\beta:\alpha\leftarrow\beta,\alpha'\leftarrow\beta'} f(v, \beta, v', \beta'),$$

then $\mathsf{val}(G \otimes H, f, g) = \langle g, (G \otimes H)f \rangle$.

Definition 2.2 (Collision Value and Symmetrized Game). *Let G be a projection game, and let f be an assignment to $\mathcal{B}$:*

$$\|Gf\| = \langle Gf, Gf\rangle^{1/2}.$$

$$\|G\| = \max_f \|Gf\|.$$

Given a projection game G, define G_{sym} to be the following game. A referee chooses $u \in U$, and then $v, v'|u$ independently, and sends v to $\mathcal{A}$ and v' to $\mathcal{B}$. $\mathcal{A}$ responds with β and $\mathcal{B}$ with β'. The referee accepts iff there exists α such that $\alpha \leftarrow \beta$ and $\alpha \leftarrow \beta'$ (the values collide).

Consider the (weighted) graph induced by the G_{sym}, i.e. V is the set of vertices. The probability we assign to (v, v') is the probability induced by first picking $u \in U$ and then $v, v' \sim u$. G is called λ-expander if the induced graph of G_{sym} is λ-expander.

Claim 2.3.

1. $\mathsf{val}(G_{sym}) = \|G\|$.
2. $\mathsf{val}(G) \leq \|G\| \leq \mathsf{val}(G)^{1/2}$.
3. $\mathsf{val}(\|G \otimes H\|) \leq \mathsf{val}(\|H\|)$.

3 Proof of the Main Result

Lemma 3.1 (Main Lemma). *Let G, H be any projection games satisfying $\mathsf{val}(G) = 1 - \epsilon$, then:*

$$\|G \otimes H\|^2 \leq (1 - c\epsilon^2) \|H\|^2.$$

For every projection game we define the following quantities:

$$\rho_G := \sup_H \frac{\|G \otimes H\|}{\|H\|},$$

$$\lambda_+(G) := \max_{h \geq 0} \frac{\|Gh\|}{\|Th\|}.$$

Where T is the following trivial game on which the referee picks $(v, v') \in V$ and accepts iff $\mathcal{A}$ answers 1.

Lemma 3.2 (Main Technical Lemma).

1. *Let G, H be any projection games, then: $\|G \otimes H\| \leq \lambda_+(G) \|H\|$.*
2. *Let G be a λ-expander projection game, then if $\lambda_+(G) > 1 - \delta$ then $\|G\| > 1 - O(\delta^2/\lambda)$.*