A simple algorithm for random colouring $G_{n,d/n}$ using $(2+\epsilon)d$ colours

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December 19, 2013

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Notation.

- *The formula $X \sim \Lambda$ means that X is a random variable with a probability distribution Λ.*
- *Let $U(M)$ stand for the uniform distribution over a set M.*
- *The set of all proper k-colourings of a graph G is denoted by $\Omega(G)$ or just Ω.*

Definition: *Let ν_1, ν_2 be two probability distributions on the set Γ. Then their **total variation distance** is $\|\nu_1 - \nu_2\| := \sup_{A \subseteq \Gamma} |\nu_1(A) - \nu_2(A)|$.*

Definition: *Let μ, ν be two distributions on a finite set Ω. Then a distribution ω on the set $\Omega \times \Omega$ is called a **coupling** of μ and ν if the following holds:*

$$(\forall x \in \Omega) \qquad \mu(x) = \sum_{y \in \Omega} \omega(x,y) \quad \& \quad \nu(x) = \sum_{y \in \Omega} \omega(y,x)$$

Theorem (Coupling lemma): *Let μ, ν be two distributions on a finite set Ω and ω their coupling:*

1. *Let $(X,Y) \sim \omega \Rightarrow \|\mu - \nu\| \leq Pr[X \neq Y]$.*
2. *There always exists a coupling ω such that $\|\mu - \nu\| = Pr[X \neq Y]$ for $(X,Y) \sim \omega$.*

Definition: *For $\sigma \in \Omega$ and some colour $q \in [k] \setminus \{\sigma_v\}$ we define the **disagreement graph** $Q_{\sigma_v,q} := (V', E')$ as the maximal induced subgraph of G such that*

$$V' := \{x \in V | \exists v - x \text{ path in } G \text{ consisting only of vertices with colours } \sigma_v, q\}$$

*We define the q-**switching** of the colouring σ as a function $H(\sigma, q) \colon \Omega \times [k] \to \Omega$ that returns the colouring obtained from σ by switching the colours σ_v and q on the vertices in V'.*

Theorem 1: *Let $G \sim G_{n,d/n}$, $\mu = U(\Omega(G))$ and μ' be the distribution of the colourings that is returned by the algorithm. For fixed $k \geq (2+\epsilon)d$, $\epsilon > 0$ and $d > d_0(\epsilon)$ with probability at least $1 - n^{-\frac{\epsilon}{90 \log d}}$ it holds that*

$$\|\mu - \mu'\| \in O(n^{-\frac{\epsilon}{90 \log d}})$$

Theorem 2: *The time complexity of the algorithm is $O(n^2)$ with probability at lest $1 - n^{-2/3}$.*

Definition: *Let G be a graph with two fixed vertices v and u, let $\sigma \in \Omega$ be a colouring of G. We call σ* ***good*** *if $\sigma_v \neq \sigma_u$, otherwise we call it* ***bad***.
For $c, q \in [k]$ we define $\Omega(c,q) := \{\sigma \in \Omega | \sigma_v = c \ \& \ \sigma_u = q\}$ and $\Omega_c := \{\sigma \in \Omega | \sigma_v = c\}$.
For $c \neq q$ we also define $S(c,c) := \{\sigma \in \Omega(c,c) | u \notin V(Q_{c,q})\}$ and $S(q,c) := \{\sigma \in \Omega(q,c) | u \notin V(Q_{q,c})\}$.

Definition: *Let $\Omega_1, \Omega_2 \subseteq \Omega$ and $\alpha \in [0,1]$. We say that Ω_1 is* ***α-isomorphic*** *to Ω_2 if there are sets $\Omega_i' \subseteq \Omega_i$ such that $|\Omega_i'| \geq (1-\alpha)|\Omega_i|$, for $i = 1, 2$, and $|\Omega_1'| = |\Omega_2'|$. We call (Ω_1', Ω_2') the* ***isomorphic pair*** *of Ω_1 and Ω_2.*
Let $h \colon \Omega_1' \to \Omega_2'$ be a bijection. Then any function $H \colon \Omega_1 \to [k]^V$ is called ***α-function*** *if $H|_{\Omega_1'} = h$.*

Lemma 1: *Assume that the sets Ω_1 and Ω_2 are α-isomorphic and $H \colon \Omega_1 \to [k]^V$ is the α-function. Let $z \sim U(\Omega_1)$, $z' = H(z)$ and let ν' be the distribution of z'. Then we have $\|U(\Omega_1) - \nu'\| \leq \alpha$.*

Lemma 2: *For any $c, q \in [k]$ with $c \neq q$ the sets $S(c,c)$ and $S(q,c)$ have the same cardinality and the q-switching function $H(\cdot, q) \colon S(c,c) \to S(q,c)$ is a bijection.*

Theorem 5: *Let $G_0, \ldots, G_r = G$ be the sequence of graphs produced by the algorithm on the input G and k. Assume that for every $i = 0, \ldots, r-1$ we have $\alpha_i \in [0,1]$ such that for any $c, q \in [k]$, $c \neq q$, the set $\Omega_i(c,c)$ is α_i-isomorphic to the set $\Omega_i(q,c)$ with α_i-function $H(\cdot, q)$. Let μ' be the distribution of the colourings that is returned by the algorithm. Then we have $\|U(\Omega(G)) - \mu'\| \leq \sum_{i=0}^{r-1} \alpha_i$.*

Corollary 3: *Let G_i be some fixed graph. For every $c, q \in [k]$ with $c \neq q$, the set $\Omega_i(c,c)$ is α-isomorphic to $\Omega_i(q,c)$ with α-function $H(\cdot, q)$ if and only if the following holds. Choose u.a.r a colouring $\sigma \in \Omega_i(c,c)$. Then $\alpha \geq \max_{q \in [k] \setminus \{c\}} \Pr[u_i \in Q_{\sigma_{v_i}, q} | G_i]$ and the analogous condition holds for a random colouring of $\Omega_i(q,c)$.*

Lemma 5: *With probability at least $1 - n^{-2/3}$ we can have the sequence $G_0, \ldots, G_r$ of subgraphs of $G_{n,d/n}$ satisfying the following three properties:*

1. *G_0 consists only of isolated vertices and simple cycles, i.e. with no common edge, each of the maximum length less than $\frac{\log n}{9 \log d}$.*
2. *In G_i the graph distance $\mathrm{dist}_{G_i}(v_i, u_i)$ is at least $\frac{\log n}{9 \log d}$.*
3. *We have $\Pr[r \geq (1 + n^{-1/3}) dn/2] \leq \exp(-n^{1/4})$.*

Theorem 7: *Take $k \geq (2+\epsilon)d$ where $\epsilon > 0$ and $d \geq d_0(\epsilon)$ are fixed. For every $i = 0, \ldots, r-1$ there is β_i such that for any $\alpha \geq \beta_i$ and any $c, q \in [k]$, $c \neq q$ the sets $\Omega_i(c,c)$ and $\Omega_i(q,c)$ are α-isomorphic and $H(\cdot, q)$ is the α-function, while*

$$E[\beta_i] \leq \frac{(40 + 8\epsilon)k}{\epsilon} n^{-(1 + \frac{\epsilon}{45 \log d})}.$$

Proposition 2: *Take $k \geq (2+\epsilon)d$ where $\epsilon > 0$ and $d \geq d_0(\epsilon)$ are fixed. Let σ be a k-colouring of G_i that is chosen u.a.r. from $(\Omega_i)_c$. For some $q \in [k] \setminus \{c\}$ we define the event $A_i :=$ "$u_i \in Q_{\sigma_{v_i}, q}$". Then we have*

$$\Pr[A_i] \leq \frac{(10 + 2\epsilon)}{\epsilon} n^{-(1 + \frac{\epsilon}{45 \log d})} \qquad i = 0, \ldots, r-1.$$