Ramsey Theory, Integer partitions and a New Proof of the Erdős-Szekeres Theorem

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ESL, EST, Partitions

Let $f(a,b)$ be the smallest integer so that every sequence of $f(a,b)$ distinct real numbers contains either an increasing sequence of length a or a decreasing sequence of length b.

Proposition 1 (Erdős-Szekeres Lemma) $f(n,n) \leq (n-1)^2+1$.

Let $g(a,b)$ be the smallest integer so that every set of $g(a,b)$ points in the plane in general position, all with distinct x-coordinates, contains either a points $p_1, p_2, \ldots, p_a$ with increasing x-coordinate so that the slopes of the segments $(p_1,p_2), (p_2,p_3), \ldots, (p_{a-1},p_a)$ are increasing, or b points such that the slopes of these segments are decreasing.

Proposition 2 (Erdős-Szekeres Theorem) $g(n,n) \leq \binom{2n-4}{n-2}+1$.

A decreasing sequence of nonnegative integers $a_1 \geq a_2 \geq \ldots$ will be called a line partition. A matrix A of nonnegative integers such that $A_{i,j} \geq A_{i+1,j}$ and $A_{i,j} \geq A_{i,j+1}$ is called a plane partition. Define a d-dimensional partition as a d-dimensional (hyper)matrix A of nonnegative integers so that the matrix is decreasing in each line, that is $A_{i_1,\ldots,i_t,\ldots,i_d} \geq A_{i_1,\ldots,i_t+1,\ldots,i_d}$ for every $i_1,\ldots,i_d$ and $1 \leq t \leq d$.

Let $P_d(n)$ be the number of $n \times \cdots \times n$ d-dimensional partitions with entries from $[n]_0$. We have

- $P_1(n) = \binom{2n}{n}$ and $P_2(n) = \prod_{1 \leq i,j,k \leq n} \frac{i+j+k-1}{i+j+k-2}$.
- $P_d(n) \leq 2^{2n^d}$ since a d-dimensional partition is composed of n^{d-1} line partitions.

Main results

Let K_N^k denote the complete k-uniform hypergraf on N ordered vertices. For a sequence of vertices $x_1 < x_2 < \cdots < x_{n+k-1}$ we say that the edges $\{x_1,\ldots,x_k\}, \{x_2,\ldots,x_{k+1}\}, \ldots, \{x_n,\ldots,x_{n+k-1}\}$ form a monotone path of length n.

Let $N_k(q,n)$ be the smallest integer N such that every coloring of the edges of K_N^k with q colors contains a monochromatic monotone path of length n.

- $f(n+1,n+1) \leq N_2(2,n) \leq n^2+1$
- $g(n+2,n+2) \leq N_3(2,n) \leq \binom{2n}{n}+1$

Theorem 1 For every $q \geq 2$ and $n \geq 2$ we have

$$N_3(q,n) = P_{q-1}(n)+1.$$

Theorem 2 For every $d \geq 1$ and $n \geq 1$ we have

$$P_d(n) \geq 2^{2n^d/3\sqrt{d+1}}.$$

Corollary 1 For every $q \geq 2$ and $n \geq 2$ we have

$$2^{2n^{q-1}/3\sqrt{q}} \leq N_3(q,n) \leq 2^{2n^{q-1}}.$$

Let $t_k(x)$ be a tower of exponents of height $k-1$ with x at the top. So $t_3(x) = 2^{2^x}$.

Theorem 3 For every $k \geq 3, q \geq 2$ and $n \geq 2$ we have

$$N_k(q,n) \leq t_{k-2}(N_3(q,n)).$$

Theorem 4 There is an absolute constant n_0 so that for every $k \geq 3, q \geq 2$ and $n \geq n_0$ we have

$$N_k(q,n) \geq t_{k-2}(N_3(q,n)/3n^q).$$

Corollary 2 For every $k \geq 3$ we have

$$N_k(2,n) = t_{k-1}((2 - o(1))n),$$

where the $o(n)$ term goes to 0 as $n \to \infty$.

Corollary 3 For every $k \geq 3, q \geq 2$ and sufficiently large n we have

$$t_{k-1}(n^{q-1}/2\sqrt{q}) \leq N_k(q,n) \leq t_{k-1}(2n^{q-1}).$$