PSD Matrix Completion and Universal Rigidity

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The main question.
Given a partial matrix (i.e., a matrix with some entries unspecified), decide whether there exist a completion s.t. the resulting matrix is positive semidefinite.

PSD matrices.
the following conditions are equivalent for positive semidefiniteness of a symmetric matrix X (denoted by $0 \preceq X$):

- $\forall x \in \mathbb{R}^n \colon x^T X x \geq 0$,
- $X = p^T p$ for some $p \in \mathbb{R}^n$,
- all eigenvalues of X are non-negative.

Semidefinite programming.
Let $\mathcal{S}^n$ denote the set of all symmetric $n \times n$ real matrices, and $\langle , \rangle$ scalar product on $\mathcal{S}^n$ defined by $\langle A, B\rangle = \mathrm{tr}(AB) = \sum_{i,j} A_{ij} B_{ij}$. By (primal) feasible region we denote any set of the form

$$\mathcal{P} = \{X \in \mathcal{S}^n \colon X \succeq 0,\ \langle A_i, X\rangle = b_i\}.$$

Together with a function $\langle C, X\rangle$ which is to be maximized, we speak of semidefinite program. Its dual is $\{\sum y_i A_i + Z = C,\ Z \succeq 0\}$ with objective function $\inf\{\sum b_i y_i\}$.

Tensegrity frameworks.
Tensegrity framework is a graph G with edges E partitioned into bars B, cables C and struts S; and an assignments of vectors $P = \{p_1, p_2, \ldots, p_n\}$ to the vertices V. Such framework is called *universally completable*, if the matrix $P^T P$ is a unique solution to the program

$$\begin{aligned}\{X \succeq 0,\ \langle E_{ij}, X\rangle &= p_i^T p_j \text{ for } (i,j) \in V \cup B,\\ \langle E_{ij}, X\rangle &\geq p_i^T p_j \text{ for } (i,j) \in S,\\ \langle E_{ij}, X\rangle &\leq p_i^T p_j \text{ for } (i,j) \in C\}.\end{aligned}$$

Sufficient condition for universal completability.
Let G, P be a tensegrity framework such that P spans $\mathbb{R}^d$. Assume there exists a matrix $Z \in \mathcal{S}^n$ satisfying

1. $Z \succeq 0$,
2. $Z_{ij} = 0$ for all $(i,j) \notin E \cup V$,
3. $Z_{ij} \geq 0$ for all cables and $Z_{ij} \leq 0$ for all struts,
4. Z has corank d,
5. $\sum_{j \in V} Z_{ij} p_j = 0$ for all $i \in 1..n$,
6. for any matrix $R \in \mathcal{S}^n$ the following holds:
$$p_i^T R p_j = 0\ \forall (i,j) \in V \cup B \cup \{(i,j) \in C \cup S \colon Z_{ij} \neq 0\} \Rightarrow R = 0.$$

Then G, P is universally completable.