Parametrized Complexity and Approximation Algorithms

by *Dániel Marx*
presented by Dušan Knop

1. Preparation

Definition 1 (NP optimization problem)**.** NP optimization problem is formally defined as a 4-tuple $(I, sol, cost, goal)$, where

- I is the set of instances;
- for as $x \in I$, $sol(x)$ is the set of feasible solutions of x. $\exists c \in \mathbb{N} \forall y \in sol(x) : |y| \leq |x|^c$ and it can be decided in polynomial time whether $y \in sol(x)$ for given x and y;
- given $x \in I$ and $y \in sol(x)$, $cost(x, y) \in \mathbb{N}$ is a polynomial-time computable;
- $goal$ is either min or max.

Let $opt(x) = goal cost(x, y') \colon y' \in sol(x)$

An algorithm has performance ratio $R(x, y)$ if it outputs $y \in sol(x)$ on $x \in I$. If

Definition 2 (c-Approximation algorithm)**.** We say that A is a c-approximation algorithm if for its performance ratio holds that $R(x, y) \leq c$.

Definition 3 (PTAS)**.** We say that a problem X admits a PTAS if $\forall \varepsilon > 0$ there isa polynomial-time $(1 + \varepsilon)$-approximation algorithm for X.

Definition 4 (EPTAS and FPTAS)**.** An efficient PTAS (EPTAS) is a PTAS with running time of the form $f(1/\varepsilon) \cdot |x|^{O(1)}$.

While s fully polynomial PTAS (FPTAS) runs in time $(1 + \varepsilon)^{O(1)} \cdot |x|^{O(1)}$.

Definition 5 (FPT-AS)**.** Is a algorithm that given $x, k, \varepsilon > 0$ produces $(1 + \varepsilon)$-approximate solution in running time $f(k, 1/\varepsilon) \cdot |x|^{O(1)}$.

Definition 6. Let $X = (I, sol, cost, goal)$ be a optimization problem. Standard FTP-approximation algorithm with ratio c is a algorithm that on input $(x, k) \in I \times \mathbb{N}$ with $OPT(x) \leq k$ for minimization (resp. $OPT(x) \geq k$ for maximization) problem and outputs $y \in sol(x)$ in time $f(k) \cdot |x|^{O(1)}$ such that $cost(x, y) \leq k \cdot c$ (resp. $cost)x, y) \geq k/c$).

2. Parametrization by instance parameter

Theorem 1. Partial Vertex Cover *admits FPT-AS with parameter k.*

3. Structural parameter

Theorem 2. Vertex Coloring *has an FPT $\frac{7}{3}$-approximation algorithm for planar $+ kv$ graphs.*

4. Parametrization by cost

5. Non-constant performance functions

Theorem 3. *If a* **maximization** *problem X has a standard FPT-approximation algorithm with performance function $\varrho(k)$, then there is a polynomial-time $\varrho'(k)$-approximation algorithm for X, for some function $\varrho'(k)$.*

6. Some examples