Independent sets in hypergraphs

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Definitions and notation

D(Usual notation.):$\mathcal{H}, \mathcal{G}$ hypergraphs. $|\mathcal{H}|$ number of vertices, $||\mathcal{H}||$ edges. $\mathcal{I}(\mathcal{H})$ is the family of all independent sets.

D:We say a uniform $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)-dense)$ if $\forall A \in \mathcal{F} : ||\mathcal{H}[A]|| \geq \varepsilon ||\mathcal{H}||$.

D:We define the max-degree of l-tuples as

$$\Delta_l(\mathcal{H}) = max\{deg_{\mathcal{H}}(T) : T \subseteq V(\mathcal{H}), |T| = l\}.$$

D:Let H be a t-uniform hypergraph with at least $t+1$ vertices. We define the t-density of H, denoted by $m_t(H)$, by

$$m_t(H) = max(\frac{||H|| - 1}{|H| - t} | H' \subseteq H, |H'| \geq t+1).$$

D:We say that H is t*-balanced* if $m_t(H') \leq m_t(H)$ for all $H' \subseteq H$.

Main theorem

T:$\forall k \in N$ and all positive c, c', ε there exists a positive constant C such that the following holds:

Let $\mathcal{H}$ be a k-uniform hypergraph and let $\mathcal{F} \subseteq 2^{V(\mathcal{H})}$ be an increasing family such that $\forall A \in \mathcal{F} : |A| \geq \varepsilon |\mathcal{H}|$. Assume also that $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$-dense and $p \in (0,1)$ is set such that $p^{k-1}||\mathcal{H}|| \geq c'|\mathcal{H}|$ and $\forall l \in [k-1]$:

$$\Delta_l(\mathcal{H}) \leq c \cdot min(p^{l-k}, p^{l-1}\frac{||\mathcal{H}||}{|\mathcal{H}|}).$$

Then there exists a family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq Cp|\mathcal{H}|}$ and functions $f : \mathcal{S} \to \overline{\mathcal{F}}$ and $g : \mathcal{I}(\mathcal{H}) \to S$ such that for every $I \in \mathcal{I}(\mathcal{H})$:

$$g(I) \subseteq I, I \setminus g(I) \subseteq f(g(I)).$$

"Roughly speaking, if $\mathcal{H}$ satisfies certain technical conditions, then each independent set I in $\mathcal{H}$ can be labeled with a certain small subset $g(I)$ in such a way thaht all sets labeled with $S \in \mathcal{S}$ are essentially contained in a single set $f(S)$ that contains very few edges of $\mathcal{H}$."

Applications

Szemerédi's theorem for sparse sets

L(Robust version of Szemerédi's theorem):

$\delta > 0, k \in [n] \exists \varepsilon > 0, \exists n_0 \forall n \geq n_0$: Every subset of $[n]$ with at least δn elements contains at least $\varepsilon n^2 k$-term APs.

T(Szemerédi's theorem for sparse sets): For every positive β and k integer, there exist constants C and n_0 such that the following holds:

For all $n \geq n_0$ if $m \geq Cn/n^{-(k-1)}$, then there are at most $\binom{\beta n}{m}$ m-subsets of $[n]$ that contain no k-term AP.

KŁR conjecture

D:Given a $p \in [0,1]$, a bipartite graph between V_1, V_2 is (ε, p)*-regular* if for every $W_1 \subseteq V_1$, $W_2 \subseteq V_2$, $|W_i| \geq \varepsilon |V_i|$, the density $d(W_1, W_2)$ satisfies

$$|d(W_1, W_2) - d(V_1, V_2)| \leq \varepsilon p.$$

D:The collection $\mathcal{G}(H, n, m, p, \varepsilon)$ is a collection of all graphs G constructed thus:

$V(G) \equiv$ partitions of vertices $V_1 \cup V_2 \cup V_{|H|}$ of n vertices. We add an (ε, p) regular pair with m edges for each edge of H.

D:A *canonical copy* of $H \equiv$ a copy of H in a member of $\mathcal{G}(H, n, m, p, \varepsilon)$ such that for each $i \in V(G)$, $f(i) \in V_i(H)$.

T(The embedding lemma):For every graph H and every positive d, there exists $\varepsilon > 0$ and an integer n_0 such that $\forall n, m, n \geq n_0, m \geq dn^2$, every $G \in \mathcal{G}(H, n, m, 1, \varepsilon)$ contains a canonical copy of H.

The problem with embedding lemma: 1 is not p. Can it be salvaged? Not entirely, but maybe only a franction of regularity-type graphs do not satisfy it:

D:$\mathcal{G}^*(H, n, m, p, \varepsilon) \equiv$ a collection of graphs in $\mathcal{G}(H, n, m, p, \varepsilon)$ which do not contain any canonical copy of H.

Q(The KŁR Conjecture):Let H be a fixed graph. Then, for any positive β, there exist positive C, n_0, ε such that $\forall n, m, n \geq n_0, m \geq \frac{Cn^2}{n^{1/m_2(H)}}$:

$$|\mathcal{G}^*(H, n, m, m/n^2, \varepsilon)| \leq \beta^m \binom{n^2}{m}^{||H||}.$$

KŁR proven for small complete graphs, cycles. One of the main results is:

T(KŁR for 2-balanced graphs):For H being 2-balanced, KŁR conjecture holds.

Proving KŁR for 2-balanced graphs

D:$\mathcal{G}(H, n_1, n_2, \ldots, n_{|H|}) \equiv$ similar to $\mathcal{G}(H, n, m, p, \varepsilon)$, only edges are complete bipartite graphs and sizes of partitions are variable.

L(Variant of the embedding lemma): Let H be a graph, $\delta : (0,1] \to (0,1)$ function. There exist positive constants α_0, ξ, N such that for every collection of integers $n_1, n_2, \ldots, n_{|H|}$, and every graph $G \in \mathcal{G}(H; n_1, n_2, \ldots, n_{|H|})$, one of the following holds:

- G contains at least $\xi n_1 n_2 \cdots n_{|H|}$ canonical copies of H,
- There exist a positive constant α with $\alpha \geq \alpha_0$, an edge $\{i, j\} \in E(H)$ and sets A_i, A_j which are of size at least $\alpha n_i, \alpha n_j$ but:

$$d_G(A_i, A_j) < \delta(\alpha).$$

L(Counting canonical copies with a non-regular pair): For each $\beta \in (0,1)$,set

$$\delta(x) = \frac{1}{4e}\left(\frac{\beta}{2}\right)^{2/x^2}.$$

Then, for every positive α_0, β, there exists a positive constant ε such that the following holds. Let $G' \subseteq K_{n,n}$ be such that there exist subsets A_1, A_2 with $min(|A_1|, |A_2|) \geq \alpha n$ and $d_G(A_1, A_2) < \delta(\alpha)$ for some $\alpha \in [\alpha_0, 1]$.

Then for every m with $0 \leq m \leq n^2$, there are at most $\beta^m \binom{n^2}{m}$ subgraphs of G' that belong to $\mathcal{G}(K_2, n, m, m/n^2, \varepsilon)$.

C(Hypergraph of copies satisfies the Scythe): Let n, t be integers with $t \geq 2$ and H be a t-balanced, t-uniform hypergraph. Set $k = ||H||$ and let $\mathcal{H}$ be the k-uniform hypergraph of copies of H in K_n^t.

Then there exist positive constants c, c' such that, letting

$$p = \frac{1}{n^{1/m_t(H)}},$$

the following holds:

- $p^{k-1}||H|| \geq c'|H|$,
- For every $l \in [k-1]$:

$$\Delta_l(\mathcal{H}) \leq c \cdot min(p^{l-k}, \frac{p^{l-1}||\mathcal{H}||}{|\mathcal{H}|}).$$

The Scythe

Given a $(i+1)$-uniform hypergraph $\mathcal{H}_{i+1}$ and an independent set $I \in \mathcal{I}(\mathcal{H}_{i+1})$ set $\mathcal{A}_{i+1}^{(0)} = \mathcal{H}_{i+1}$ and let $\mathcal{H}_i^{(0)}$ be the empty hypergraph on the vertex set $V(\mathcal{H})$. For $j = 0 \ldots b-1$, do the following:

- If $I \cap V(\mathcal{A}_{i+1}^{(j)}) = \emptyset$, set $\mathcal{H}_i = \mathcal{H}_i^{(0)}$, $\mathcal{A}_i = \emptyset$, $B_i = \{u_0, \ldots, u_{j-1}\}$ and stop.
- Let u_j be the first vertex of I in the max-degree order on $V(\mathcal{A}_{i+1}^{(j)}$.
- Let $\mathcal{H}_i^{j+1}$ be the hypergraph on the vertex set $V(\mathcal{H})$ defined by:

$$.$$

- Let $\mathcal{A}_{i+1}^{j+1}$ be the hypergraph on the vertex set $V(\mathcal{A}_{i+1}^{(j)}) \setminus u_{1 \ldots j}$ defined by:

$$.$$

Finally, set $\mathcal{H}_i = \mathcal{H}_i^{(b)}$, $\mathcal{A}_i = V(\mathcal{A}_{i+1}^{(b)})$ and $B_i = u_{1 \ldots b-1}$.