COUNTING PLANE GRAPHS: PERFECT MATCHINGS, SPANNING CYCLES, AND KASTELEYN'S TECHNIQUE

MICHA SHARIR, ADAM SHEFFER, EMO WELZL

Presented by: Marek Eliáš

Definition (sc(N))**.** For a set S of points in the plane, we denote by $\mathcal{C}(S)$ the set of all crossing-free straight-edge spanning cycles of S, and put $\mathrm{sc}(S) := |\mathcal{C}(S)|$. Moreover, we let $\mathrm{sc}(N) = \max_{|S|=N} \mathrm{sc}(S)$.

Definition (tr(N))**.** For a set S of points in the plane, we denote by $\mathcal{T}(S)$ the set of all triangulations of S, and put $\mathrm{tr}(S) := |\mathcal{T}(S)|$. Moreover, we let $\mathrm{tr}(N) = \max_{|S|=N} \mathrm{tr}(S)$.

Definition.
Hull edges/vertices: those which are part of the boundary of the convex hull of S
Interior edges/vertices: those which are not part of the boundary of $\mathrm{conv}(S)$
h: number of hull vertices
n: number of interior vertices ($n := N - h$)

Theorem (Already known bounds)**.**
$\mathrm{tr}(N) < 30^N$
$\mathrm{sc}(N) < 30^{N/4} \cdot \mathrm{tr}(S) \approx 2.3404^N \cdot \mathrm{tr}(S)$
$\Rightarrow \mathrm{sc}(N) = O(68.664^N)$

Definition ($\mathrm{sc}_\Delta(N)$)**.**
$\mathcal{C}(T)$: set of spanning cycles contained in a triangulation T
$\mathrm{sc}_\Delta(N) := \max_{|S|=N, T\in\mathcal{T}} |\mathcal{C}(T)|$ i. e. maximal number of plane spanning cycles that can be contained in any fixed triangulation of a set of N points in the plane.

Definition (Support)**.** Given a plane edge graph G embedded over a set S of points in the plane, we say that G has a support of x if G is contained in exactly x triangulations of S; we write $\mathrm{supp}(G) = x$.

$$\mathrm{sc}(S) = \sum_{T\in\mathbb{T}(S)} \sum_{C\in\mathcal{C}(T)} \frac{1}{\mathrm{supp}(C)} \tag{1}$$

Definition (ps-flippable edge)**.** Given a triangulation T, we say that a subset F of its edges is a set of ps-flippable edges, if F are diagonals of interior-disjoint convex polygons whose boundaries are also parts of T.

Lemma (L2.1)**.** *Every triangulation T over a set of N points in the plane contains a set F of $N/2 - 2$ ps-flippable edges. Also, there are triangulations with no larger sets of ps-flippable edges.*

Lemma (L2.2)**.** *Consider a triangulation T, a set F of $N/2 - 2$ ps-flippable edges in T, and a graph $G \subseteq T$. If G does not contain j edges from F, then* $\mathrm{supp}(G) \geq 2^j$.

Definition (Convex decomposition)**.** A conv. decomposition of a point set S is a crossing-free straight-edge graph D on S such that (i) D includes all the hull edges, (ii) each bounded face of D is a convex polygon, (iii) no point of S is isolated in D.

Lemma (L2.3)**.** *Let S be a set of points in the plane and let G be a crossing-free straight-edge graph over S that contains all the edges of the convex hull of S. Then G is a convex decomposition of S if and only if every interior vertex of S is valid with respect to G.*

Date: 2014-01-02.

Lemma (L2.4)**.** *Let $c > 1$ be a constant such that every set S of an even number of points in the plane satisfies* $\mathrm{sc}(S) = O(c^{|S|})$*. Then* $\mathrm{sc}(S) = O(c^{|S|})$ *also holds for sets S of an odd number of points.*

Definition ($\mathrm{pm}(N)$)**.** For a set S of points in the plane, we denote by $\mathcal{M}(S)$ the set of all perfect matchings of S, and put $\mathrm{pm}(S) := |\mathcal{M}(S)|$. Moreover, we let $\mathrm{pm}(N) = \max_{|S|=N} \mathrm{pm}(S)$. $\mathcal{M}(G)$ is a set of perfect matchings contained in G.

Theorem (T3.1)**.** *For any set S of N points in the plane,*

$$\mathrm{sc}(S) = O(12^{N/4}) \cdot \mathrm{tr}(S) = O(1.8613^N) \cdot \mathrm{tr}(S)$$

Theorem (Kasteleyn's enhanced theorem)**.** *Every planar graph G can be oriented onto some digraph $\vec{G}$ such that, for any real-valued weight function μ on its edges, we have*

$$\left(\sum_{M \in \mathcal{M}(G)} \mu(M) \right)^2 = |\det(B_{\vec{G},\mu})|$$

Theorem (T4.1)**.** *For any set S of N points in the plane,*

$$\mathrm{pm}(S) \le 8 \cdot (3/2)^{N/4} \cdot \mathrm{tr}(S) = O(1.1067^N) \cdot \mathrm{tr}(S)$$

Lemma (L5.1)**.** *Let T be a triangulation over a set S of $N \ge 6$ points in the plane, such that N is even and S has a triangular convex hull; also, let $v_3(T) = tN$ be the number of interior vertices of degree 3 in T. Then*

$$\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathrm{supp}(C)} < 8 \left(\frac{3}{2^t} \left(\frac{(2-t)(2-t/2)}{(1-t)^2} \right)^{1-t} \right)^{N/4}$$

Lemma (L5.2)**.** *Let c_{gon} be the maximum real number satisfying the following property: Every simple polygon P that has a triangulation T_P with k flippable and with $l \le k$ of these diagonals forming a ps-flippable set, has at least $2^l c_{gon}^{k-l}$ triangulations. Then $x \le c_{gon} \le 5/4$ with $x \approx 1.17965$ the unique real root of the polynomial $1 + 4x^2 - 4x^3$.*

Lemma (L5.3)**.** *Consider a triangulation T with the number of flippable edges* $\mathrm{flip}(T) = N/2 - 3 + \kappa N$*, for some $\kappa \ge 0$, and let x be the unique real root of the polynomial $1 + 4x^2 - 4x^3$. Then*

$$\sum_{C \in \mathcal{C}(T)} \frac{1}{\mathrm{supp}(C)} < 8 \left(\frac{(3 + (\gamma^2 - 1)(\kappa + 1/2))(4 + (x^2 - 1)\kappa)}{x^{4\kappa}} \right)^{N/4},$$

where $\gamma = x \cdot e^{-\frac{x^2-1}{4(4+(x^2-1)\kappa)}}$.

Lemma (L5.4)**.** *Let $c > 1$ be a constant such that every set S of an even number of points in the plane and a triangular convex hull satisfies* $\mathrm{sc}(S) = O(c^{|S|})$*. Then* $\mathrm{sc}(S) = O(c^{|S|})$ *also holds for every other finite point-set S in the plane.*

Theorem (T5.5)**.** *For any set S of N points in the plane,*

$$\mathrm{sc}(S) = O(10.9247^{N/4}) \cdot \mathrm{tr}(S) = O(1.8181^N) \cdot \mathrm{tr}(S)$$

Corollary (C5.6)**.** $\mathrm{sc}(N) = O(54.5430^N)$.