Computing the Distance between Piecewise-Linear Bivariate Functions

by Guillaume Moroz and Boris Aronov

We will show, how to compute the distance of two bivariate piecewise linear functions f and g given on a common domain M, but defined piecewise over different triangulations, in $\mathcal{O}(n\log^4 n)$ time.

The Problem and Notation

- functions $f(x,y)$, $g(x,y) : M \to \mathbb{R}$
- two different triangulations of M: T_f, T_g
- distance: L_2-norm i.e. $\|f-g\|_2 = (\iint\limits_M (f(x,y)-g(x,y))^2 \mathrm{d}x\mathrm{d}y)^{\frac{1}{2}}$
- the problem is efficient computation of $\iint\limits_M f(x,y)g(x,y)\mathrm{d}x\mathrm{d}y$

Integrating over Convex Polygons

- convex polygon C, p a vertex of C
- lines supporting the edges of C incident to p:
  - lower: $L(p,C) : y = y_l(p,C) + s_l(p,C)x$
  - upper: $U(p,C) : y = y_u(p,C) + s_u(p,C)x$
-
$$\delta(p,C) = \begin{cases} -1 \text{ if } C \text{ is above/below both } L(p,C) \text{ and } U(p,C) \\ +1 \text{ if } C \text{ is above/below } L(p,C) \text{ and below/above } U(p,C) \end{cases}$$
- denote $\mathcal{T}(p,C,h) = \delta(p,C) \int\limits_{x=0}^{x_p} \int\limits_{L(p,C)}^{U(p,C)} h(x,y)\mathrm{d}x\mathrm{d}y$

Counting the integral over a polygon C:

Theorem. 2.1. *Let C be a convex polygon with vertices $p_1, p_2, \ldots p_k$ and $h(x,y)$ a bivariate function. Then*

$$\iint\limits_C h(x,y)\mathrm{d}x\mathrm{d}y = \sum_{j=1}^{k} \mathcal{T}(p_j, C, h).$$

Counting over all M:

Theorem. 2.2.

$$\iint\limits_M h(x,y)\mathrm{d}x\mathrm{d}y = \sum_{cell\ C\in\mathcal{C}} \iint\limits_C h(x,y)\mathrm{d}x\mathrm{d}y = \sum_{C \text{ adjacent to } p} \sum_{cell C\in\mathcal{C}} \mathcal{T}(p,C,h)$$

Sum of the integrals (of functions piecewise linear, as supposed) in fact can be expressed as a polynomial:

Theorem. 2.3. *Let p be an intersection point of $L(p,C)$ and $U(p,C)$, then $x_p = -\frac{y_u - y_l}{s_u - s_l}$ and*

$$\int\limits_{x=0}^{x_p} \int\limits_{L(p,C)}^{U(p,C)} x^i y^j \mathrm{d}x\mathrm{d}y = \frac{P_{i,j}(y_l, y_u, s_l, s_u)}{(s_u - s_l)^{i+j+1}},$$

where $P_{i,j}$ is a polynomial of total degree $i + 2j + 2$.

When counting the sums on the right, we go through two sets of points: the original points (lets denote the set Σ_v) of triangulations T_f and T_g and the intersecting points (Σ_e) that appear in the overlay of the two triangulations.

- point location can be done in $\mathcal{O}(n\log n)$ time
- complexity of computing the integral over points from Σ_v depends on the degrees i.e. can be done in $\mathcal{O}(n)$ time

Bipartite Clique Decomposition

Having triangulations T_f and T_g given by edges E_f and E_g, we can find a family $\mathcal{F} = \{(R_1, B_1), \ldots (R_u, B_u)\}$, $R_k \subset E_f$, $B_k \subset E_g$ fulfilling:

- every edge in R_k intersects every edge in B_k
- every edge in R_k has lower slope than every edge in B_k, or vice versa
- for every intersecting pair (e_r, e_b), $e_r \in E_f, e_b \in E_g$ there $\exists!$ k such that $e_r \in R_k, e_b \in B_k$
- no such k exists for non-intersecting pairs (e_r, e_b)
- $\sum\limits_k (|R_k| + |B_k|) = \mathcal{O}(n\log^2 n)$

Effective Computation of the Integral over Σ_e

We consider the integral over a bipartite clique decomposition, therefore let $(R, B) = (R_k, B_k)$, we have:

$$\sum_{\substack{p=e_r\cap e_b \\ (e_r,e_b)\in R\times B}} \sum_{C \text{ adjacent to } p} \mathcal{T}(p, C, h_C) = \sum_{e_r\in R} \sum_{e_b\in B} \sum_{\substack{C \text{ adjacent} \\ \text{to } p=e_r\cap e_b}} \mathcal{T}(p, C, h_C).$$

Each point of Σ_e lies on the boundary of four cells. Each of them can be computed extra.
This integral can be expressed as a polynomial with variables the coefficients of the edges that create borders of the cell:

$$\sum_{e_r\in R} \sum_{e_b\in B} \int\limits_{x=0}^{x_p} \int\limits_{L(p,C)}^{U(p,C)} v(e_r)w(e_b)x^i y^j \mathrm{d}x\mathrm{d}y = \sum_{e_r\in R} \sum_{e_b\in B} \frac{v(e_r)w(e_b)P_{i,j}(y(e_r), y(e_b), s(e_r), s(e_b))}{(s(e_b) - s(e_s))^{i+j+1}}.$$

Fast Multi-point Evaluation

We will use symbolic variables for edges of R and numerical values for edges of B and so compute the value of the rational function above. I.e. the last sum can be written as:

$$F(X, Y, V) = \sum_{e_r\in R} \sum_{e_b\in B} \frac{Vw(e_b)P_{i,j}(Y, y(e_b), X, s(e_b))}{(X - s(e_b))^{i+j+1}}.$$

This is a rational function of three variables with X in the denominator, of order $i + j + 1$. Using theorem **5.1.** we can compute the coefficients of the polynomial, and in time $\mathcal{O}(n\log^2 n)$ express it in form:

$$\frac{N(X, Y)V}{D(X)},$$

where both numerator and denominator are of small degrees.

And, that can be rewritten in form $\sum\limits_{e_r\in R} \frac{N(s(e_r),y(e_r))v(e_r)}{D(s(e_r))}$.

Using theorem **5.2.** we can compute simultaneously all the terms of this sum and that in only $\mathcal{O}(n\log^2 n)$ operations.

For each pair (R_k, B_k) of the decomposition we can compute the sum in $\mathcal{O}((|B_k| + |R_k|)\log^2(|B_k| + |R_k|))$, $\sum\limits_k (|B_k| + |R_k|) = \mathcal{O}(n\log^2 n)$.

I.e. $\mathcal{O}((|B_k| + |R_k|)\log^2(|B_k| + |R_k|)) \leq \mathcal{O}(|B_k| + |R_k|)\log^2 n = \mathcal{O}(n\log^4 n)$.

Theorem. 5.1. *Let u and v be two functions from the edges of B to $\mathbb{R}$, suppose $|B| \leq n$. Then $\sum\limits_{e\in B} \frac{e}{(X-v(e))^d}$ can be expressed in form $\frac{N(X)}{D(X)}$, where $N(X)$ and $D(X)$ are polynomials of degree at most $(n-1)d$ and nd. Their coefficients can be computed in $\mathcal{O}(\mathcal{M}(nd)logn)$ time, where $\mathcal{M}(q) = \mathcal{O}(qlogq)$ is the cost of multiplication of two univariate polynomials of degree at most q.*

Theorem. 5.2. *Let $P(X_0, X_1, \ldots X_r)$ be a polynomial of degree n in X_0, and d in $X_1, \ldots, X_r$, Let E be a set of at most n points of $\mathbb{R}^{r+1}$. The values of P at the points of E can be simultaneously computed in $\mathcal{O}(\binom{d+r}{r}\mathcal{M}(n)logn)$ time.*