Clique versus Independent Set

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Some definitions

Definition 1 (CS-separator)**.** A family F of m cuts such that for every disjoint clique and stable set, there is a cut in F that separates the clique and the stable set is called a CS-separator of size m.

Definition 2 (Hypergraph transversality)**.** Let H be a hypergraph. We define transversality $\tau(H)$ as the minimum cardinality of a subset of vertices intersecting each hyperedge. And corresponds to a solution of the following integer program:

$$
\begin{aligned}
\text{Minimize: } & \sum_{x \in V} w(x) & \\
\text{Subject to: } & w(x) \in \{0,1\} & \forall x \in V \\
& \sum_{x \in e} w(x) \geq 1 & \forall e \in E
\end{aligned}
$$

The fractional hypergraph transversality τ^* corresponds to a solution of a relaxed linear program.

Definition 3 (VC-dimension)**.** The VC-dimension of a hypergraph is the maximum cardinality of a set of vertices A such that for every $B \subseteq A$ there is an edge e so that $e \cap A = B$.

Definition 4 (Bipartite packing)**.** The bipartite packing $bp(G)$ of a graph G is the minimum number of edge-disjoint complete bipartite graphs needed to partition the edges of G.

The oriented bipartite packing $bp_{OR}(G)$ of a non-oriented graph G is the minimum number of oriented complete bipartite graphs such that each edge is covered by an arc in at least one direction (it can be in both directions), but it cannot be covered twice in the same direction.

Definition 5 (Fooling set)**.** A fooling set is a set F of b-inputs (i.e. for all $(x, y) \in F$ $f(x, y) = b$) of M such that for all $(x, y), (x', y') \in F$, $f(x', y) \neq b$ or $f(x, y') \neq b$.

Conjecture 1. *There is a polynomial Q, such that for every graph G on n vertices, there is a CS-separator of size at most $Q(n)$.*

Proposition 1. *Conjecture 1 holds if and only if a polynomial family F of cuts separates all the maximal (in the sense of inclusion) cliques from the maximal stable sets that do not intersect.*

The case of split-free graphs split graph can be decomposed to a clique and a stable set. For a graph Γ define a class of graphs C_Γ as the class of all graph without Γ as a induced subgraph.

Lemma 1. *Every hypergraph H with VC-dimension d satisfies $\tau(H) \leq 16d\tau^*(H)\log(d\tau^*(H))$.*

Theorem 1. *Let Γ be a fixed split graph. Then the Clique-Stable Set conjecture is verified on C_Γ.*

Let (K, S) be a pair of maximal clique and stable set. We build H a hypergraph with vertex set K. For all $x \in S$, build the hyperedge $K \setminus N_G(x)$.

Symmetrically, build H' a hypergraph with vertex set S. For all $x \in K$, build the hyperedge S NG(x).

To begin with, let us introduce an auxiliary oriented graph B with vertex set $K \cup S$. For all $x \in K$ and $y \in S$, put the arc xy if $xy \in E$, and put the arc yx otherwise.

Lemma 2. *In B, there exists:*

1. *either a weight function $w : K \to \mathbb{R}^+$ such that $w(K) = 2$ and $\forall x \in S, w(N^+(x)) \geq 1$.*
2. *or a weight function $w : S \to \mathbb{R}^+$ such that $w(S) = 2$ and $\forall x \in K, w(N^+(x)) \geq 1$.*

Lemma 3. *The hypergraph H has fractional transversality $\tau^* \leq 2$.*

Lemma 4. *H has VC-dimension bounded by $2\varphi - 1$.*

The case of $P_k, \overline{P_k}$-free graphs

Theorem 2. *Let $k > 0$. The Clique-Stable set conjecture is verified on the class of $P_k, \overline{P_k}$-free graphs.*

Theorem 3. *For every k, there is a constant $t_k > 0$, such that every graph $G \in C_k$ contains two subsets of vertices V_1 and V_2, each of size at least $t_k \cdot n$, such that V_1 and V_2 are completely adjacent or completely non-adjacent.*

Bipartite packing and graph coloring

Conjecture 2 (Polynomial Alon-Saks-Seymour Conjecture)**.** *There exists a polynomial P such that for every G, $\chi(G) \leq P(bp(G))$.*

Conjecture 3 (Oriented Alon-Saks-Seymour Conjecture)**.** *There exists a polynomial P such that for every G, $\chi(G) \leq P(bp_{OR}(G))$.*

Theorem 4. *Let $m, n \in \mathbb{N}^*$. There exists a fooling set C of size m on some graph on n vertices if and only if $bp_{OR}(K_m) \leq n$.*

Theorem 5. *The oriented Alon-Saks-Seymour conjecture is verified if and only if the Clique-Stable Set separation conjecture is verified.*