How to Play Unique Games Using Embeddings

by *Eden Chlamtac, Konstantin Makarychev, Yury Makarychev*
presented by Dušan Knop

Definition 1 (Unique games conjecture)**.** Given a constraint graph G = (V,E) and a set of permutations π_{uv} on $[k]$ (for all edges (u,v)), assign a value (state) x_u from the set $[k]$ to each vertex u so as to satisfy the maximum number of constraints of the form $\pi_{uv}(x_u) = x_v$.

Definition 2. Let X be an ℓ_2^2 space. We say that a distribution over subsets of X is an m-orthogonal separator of X with distortion D and probability scale α if the following conditions hold for $S \subset X$ chosen according to this distribution:

1. For all u in X, $Pr(u \in S) = \alpha\|u\|^2$.
2. For all orthogonal vectors u and v in X, $Pr(u \in S \text{ and } v \in S) \leq \frac{\min(Pr(u \in S), Pr(v \in S))}{m}$. Note that the right hand side is at most $\alpha \cdot \frac{\|u\|^2 + \|v\|^2}{2m}$.
3. For all u and v in X, $Pr(I_S(u) \neq I_S(v)) \leq \alpha D \|u - v\|^2$, where I_S is the indicator (characteristic) function of the set S.

Producing orthogonal separators We will proceed in three steps:

1. we transform the set X into a set of functions in $L_2[0,1]$, so that the image of every non-zero vector is a function with L_2 norm 1
2. we embed the transformed set into the unit sphere in l_1 (l_2) using slightly modified previously known algorithms
3. we boost the probability that orthogonal vectors are separated and then we recover the original lengths of all vectors and get rid of the $1/\max(\|u\|, \|v\|)$ term in the distortion

1. Normalization

$$\varphi(u)(t) = \begin{cases} u, \text{ if } t \leq 1/\|u\|^2 \\ 0, \text{ otherwise} \end{cases}$$

Lemma 1. *Let $X \subset \mathbb{R}^d$ be an l_2^2 metric space containing the zero vector. Then*

1. *The image $\varphi(X)$ satisfies triangle inequalities in L_2^2: $\forall u, v, w \in X \|\varphi(u) - \varphi(v)\|^2 + \|\varphi(v) - \varphi(w)\|^2 \geq \|\varphi(u) - \varphi(w)\|^2$.*
2. *For all vectors u and v in X, $\langle \varphi(u), \varphi(v) \rangle = \frac{\langle u, v \rangle}{\max(\|u\|^2, \|v\|^2)}$.*
3. *For all non-zero vectors u in X, $\|\varphi(u)\|^2 = 1$.*
4. *For all orthogonal u and v in X, the images $\varphi(u)$ and $\varphi(v)$ are also orthogonal.*
5. *For all non-zero vectors u and v in X, $\|\varphi(v) - \varphi(u)\|^2 \leq \frac{\|v-u\|^2}{\max(\|u\|^2, \|v\|^2)}$.*

2. Embedding into ℓ_1 We will use a modification of this well-known theorem:

Theorem 1 (Arora, Lee and Naor)**.** *There exist constants $C \geq 1$ and $0 < p < 1/2$ such that for every finite ℓ_2^2 space X with distance $d(u,v) = \|u - v\|^2$ and every $\Delta > 0$, the following holds. There exists a distribution μ over subsets $U \subset X$ such that for every $u, v \in X$ with $d(u,v) \geq \Delta$, $\mu[U\colon u \in U$ and $d(v, U) \geq \frac{\Delta}{C\sqrt{\log|X|}}] \geq p$.*

Corollary 1. *There exists an efficient algorithm that, given an ℓ_2^2 space X, generates random subsets Y such that the following conditions hold.*

1. *For every u and v in X, $Pr(I_Y(u) \neq I_Y(v)) \leq D\|u-v\|^2$.*
2. *For every u and v s.t. $\|u-v\| \geq 1$, $Pr(I_Y(u) \neq IY(v)) \geq 2p$,*

where $D = O(\sqrt{log|X|})$.

Approximation algorithm

1. Solve the SDP.
2. Mark all vertices as unprocessed.
3. while (there are unprocessed vertices)
   (a) Produce an m-orthogonal separator S with distortion D and probability scale α, where $m = 4k$ and $D = O(\sqrt{\log n \log m})$.
   (b) For all unprocessed vertices u:
      - Let $S_u = \{i \colon u_i \in S\}$.
      - If S_u contains exactly one element i, then assign the state i to u, and mark the vertex u as processed.
4. If the algorithm performs more than n/α iterations, assign arbitrary values to any remaining vertices (note that $\alpha \geq 1/poly(k)$).

Semidefinite relaxation For each vertex u and each state i we introduce a vector u_i. The intended integer solution is as follows. For every vector u_i set $u_i = 1$ if vertex u is assigned state i, otherwise let $u_i = 0$. Thus for a fixed u, only one u_i is not equal to zero. To model this property in the SDP we add the constraint that u_i and u_j are orthogonal for all $i \neq j$ and u; and the constraint $\|u_1\| + \cdots + \|u_k\| = 1$ for all u. We also add some triangle inequality constraints.

In the integer solution, if the Unique Game constraint between u and v is satisfied, then $u_i = v_{\pi_{uv}(i)}$ for all $i \in [k]$. On the other hand if the constraint is not satisfied then the equality $u_i = v_{\pi_{uv}(i)}$ is violated for exactly two values of i. Thus the expression $\varepsilon_{uv} = \frac{1}{2}\sum_{i=1}^{k}(u_i - v_{\pi_{uv}(i)})^2$ is equal to 0, if the constraint is satisfied and 1, otherwise.

minimize $\frac{1}{2}\sum_{uv \in E}\sum_{i \in [k]}\|u_i - v_{\pi_{uv}(i)}\|^2$ subject to

$$u \in V \forall i, j \in [k], i \neq j \qquad \langle u_i, u_j \rangle = 0 \tag{1}$$

$$\forall u \in V \qquad \textstyle\sum_{i \in [k]}\|u_i\|^2 = 1 \tag{2}$$

$$\forall u, v, w \in V \forall i, j, l \in [k] \quad \|u_i - w_l\|^2 \leq \|u_i - v_j\|^2 + \|v_j - w_l\|^2 \tag{3}$$

$$\forall u, v \in V \forall i, j \in [k] \qquad \|u_i - v_j\|^2 \leq \|u_i\|^2 + \|v_j\|^2 \tag{4}$$

$$\forall u, v \in V \forall i, j \in [k] \qquad \|u_i\|^2 \leq \|u_i - v_j\|^2 + \|v_j\|^2 \tag{5}$$

Lemma 2. *There is an algorithm which satisfies the constraint between vertices u and v with probability $1 - O(D\epsilon_{uv})$, where ε_{uv} is the SDP contribution of the term corresponding to the edge (u, v): $\varepsilon_{uv} = \frac{1}{2}\sum_{i \in [k]}\|u_i - v_{\pi_{uv}(i)}\|$.*

Theorem 2. *There exists a randomized polynomial time algorithm that, given an ℓ_2^2 space X containing 0 and a parameter m, returns an m-orthogonal separator of X with distortion $D = O(plog|X|logm)$ and probability scale $\alpha \geq 1/poly(m)$.*