Two extensions of Ramsey's theorem

by *David Conlon, Jacob Fox, Benny Sudakov*

presented by Martin Balko

Let $r(k)$ be the minimum n such that in every 2-coloring of the edges of the complete graph K_n there is a monochromatic K_k. Ramsey's theorem states that $r(k)$ exists for all k. Classical results of Erdős and Szekeres give $2^{k/2} \leq r(k) \leq 2^{2k}$ for $k \geq 2$.

The authors consider two strengthenings of Ramsey's theorem. Common to the proofs of both results is a probabilistic argument known as dependent random choice.

Dependent Random Choice

Every dense graph contains a large vertex subset U such that every small subset of U has many common neighbors.

Lemma 1. *Suppose $p > 0$ and m, s, t, N_1, N_2 are positive integers satisfying $\binom{N_1}{s}(m/N_2)^t \leq p^t N_1/2$. If $G = (V_1, V_2, E)$ is a bipartite graph with $|V_i| = N_i$ for $i = 1, 2$ and at least pN_1N_2 edges, then G has a vertex subset $U \subset V_1$ such that $|U| \geq p^t N_1/2$ and every s vertices in U have at least m common neighbors.*

1 Ramsey's theorem with skewed vertex distribution

Definitions:

- The *weight $w(S)$ of a finite set S* of integers greater than one is defined as $w(S) = \sum_{s \in S} \frac{1}{\log s}$.
- For a red-blue edge-coloring c of the complete graph on $\{2, \ldots, n\}$, let $f(c)$ be the maximum weight $w(S)$ over all $S \subset \{2, \ldots, n\}$ which form a monochromatic clique in coloring c.
- For $n \geq 2$, let $f(n)$ be the minimum of $f(c)$ over all red-blue edge-colorings c of the edges of the complete graph on $\{2, \ldots, n\}$.

In 1981 Erdős conjectured that $f(n)$ tends to infinity and asked for an accurate estimate of $f(n)$.

Known:

- $f(n) = \Omega\left(\frac{\log\log\log\log n}{\log\log\log\log\log n}\right)$, $f(n) = O(\log\log\log n)$ [Rödl, 2003]

Theorem 1. *For n sufficiently large, every red-blue edge-coloring of the edges of the complete graph on the interval $\{2, \ldots, n\}$ contains a monochromatic clique with vertex set S such that*

$$\sum_{s \in S} \frac{1}{\log s} \geq 2^{-8} \log\log\log n.$$

Hence, $f(n) = \Theta(\log\log\log n)$.

Lemma 2. *Suppose that the edges of K_n have been two-colored in red and blue and that each vertex v has been given positive weights r_v and b_v satisfying $b_v \geq \ln(4/r_v)$ if $r_v \leq b_v$ and $r_v \geq \ln(4/b_v)$ if $b_v \leq r_v$. Then there exists either a red clique K for which $\sum_{v \in K} r_v \geq \frac{1}{2}\ln n$ or a blue clique L for which $\sum_{v \in L} b_v \geq \frac{1}{2}\ln n$.*

Additional Remarks:

- If the limit $\lim_{n\to\infty} \frac{\log r(n)}{n}$ exists, denote it by c_0. We know that $\frac{1}{2} \leq c_0 \leq 2$.

  Conjecture 1. *We have $f(n) = \left(c_0^{-2} + o(1)\right)\log\log\log n$.*

  The construction of Rödl can be modified to obtain $f(n) \leq \left(c_0^{-2} + o(1)\right)\log\log\log n$ and a modification of the proof of Theorem 1 gives $f(n) \geq \left(\frac{1}{4} - o(1)\right)\log\log\log n$.

- Can we find cliques of large weight for other weight functions? Let $w(i)$ be a weight function defined on all positive integers $n \geq a$ and let $f(n, w)$ be the minimum over all 2-colorings of $\{a, \ldots, n\}$ of the maximum weight of a monochromatic clique.

  Theorem 2. *Let $\log_{(i)}(x)$ be the iterated logarithm given by $\log_{(0)}(x) = x$ and, for $i \geq 1$, $\log_{(i)}(x) = \log\left(\log_{(i-1)}(x)\right)$. Let $w_s(i) = 1/\prod_{j=1}^{s} \log_{(2j-1)} i$. Then $f(n, w_s) = \Theta\left(\log_{(2s+1)} n\right)$. However, letting $w'_s(i) = w_s(i) / \left(\log_{(2s-1)} i\right)^{\epsilon}$ for any fixed $\epsilon > 0$, then $f(n, w'_s)$ converges.*

- Unlike the graph case, there are colorings for which the maximum weight of a monochromatic clique is bounded in the case of 3-uniform hypergraphs. The analogue of Erdős' conjecture for 3-colorings of graphs also does not hold.

2 Ramsey's theorem with fixed order type

Jouko Väänänen asked whether, for any positive integers k and q and any permutation π of $[k-1]$, there is a positive integer R such that for any q-coloring of the edges of the complete graph on vertex set $[R]$ there is a monochromatic K_k with vertices $a_1 < \ldots < a_k$ satisfying

$$a_{\pi(1)+1} - a_{\pi(1)} > a_{\pi(2)+1} - a_{\pi(2)} > \ldots > a_{\pi(k-1)+1} - a_{\pi(k-1)}.$$

The least such integer is denoted by $R_\pi(k; q)$. We let $R(k; q) = \max_\pi R_\pi(k; q)$.

Known:

- The question was positively answered by Noga Alon (a weak bound on $R(k; q)$) and, independently, by Erdős, Hajnal and Pach in 1997 (no bound on $R(k; q)$).
- Alon and Spencer showed that $R(k; q)$ should grow exponentially in k for monotone sequences.
- The double-exponential upper bound $R(k; q) \leq 2^{\left(q(k+1)^3\right)^{qk}}$ holds [Shelah, 1997].

Theorem 3. *For any positive integers k and q, $R(k; q) \leq 2^{k^{20q}}$ holds.*

Definitions:

- An *interval* I of integers is a set of consecutive integers. Let $S \subset \mathbb{Z}$ be nonempty. The *density $d_I(S)$ of S with respect to an interval I* with $S \subset I$ is $|S| / |I|$.
- An ordered pair (T_1, T_2) of sets of integers is *separated* if, for $j = 1, 2$,

  $$\min(T_2) - \max(T_1) > \max(T_j) - \min(T_j).$$

- Let G be a graph on a subset of integers, J be an interval, and $S \subset J \cap V(G)$. For $0 < \alpha, \beta, \gamma, \delta, p < 1$, we say that G is *$(\alpha, \beta, \gamma, \delta, p)$-heavy with respect to S* if for all $S' \subset S$ for which there is an interval J' with $S' \subset J'$, $d_{J'}(S') \geq \delta d_J(S)$, and $|S'| \geq \gamma |S|$, there are $T_1, T_2 \subset S'$ and, for $j = 1, 2$, intervals I_j with $T_j \subset I_j$ such that (T_1, T_2) is a separated pair, $d_{I_j}(T_j) \geq \alpha d_{J'}(S')$, $|I_j| \geq \beta |S'|$ and the edge density of G across T_1, T_2 is at least p.
- Let $\phi\colon [h-1] \to [k-1]$ be an injective function, $0 < \eta < 1$ and $r \in \mathbb{N}$. A *clique of type (ϕ, η, r)* consists of h pairwise adjacent vertices $a_1, \ldots, a_h$ such that $a_{i+1} - a_i \in \left[\eta^{\phi(i)} r, \eta^{\phi(i)-1} r\right)$ for $i \in [h-1]$.

Lemma 3. *Suppose G is a graph on a subset of the integers, J is an interval, $S \subset J \cap V(G)$, $\phi\colon [h-1] \to [k-1]$ is an injective function, $0 < \alpha, \beta, \gamma, \delta, p < 1$, and $r \in \mathbb{N}$. Let $t = 2\sqrt{k \log_{1/p} |S|}$, $\epsilon = p^t/2$, $\lambda = \left(\frac{\epsilon\alpha}{4}\right)^{2h}$, and $\kappa = \lambda\beta d_J(S)^2 \eta^k r$. Provided that $\kappa \geq h$, $|J| \geq r$, $\eta \leq \beta\lambda d_J(S)^2$, $\delta \leq \lambda$, and $\gamma |S| \leq \kappa$, the following holds. If G is $(\alpha, \beta, \gamma, \delta, p)$-heavy with respect to S then there is a clique in G of type (ϕ, η, r).*

Additional Remarks:

- The natural hypergraph analogue of Väänänen's question fails.
- For certain π there exist 2-edge-colorings of the complete graph on positive integers in which none of the sequences $a_1 < \ldots < a_k$ satisfying

  $$a_{\pi(1)+2} - a_{\pi(1)} > a_{\pi(2)+2} - a_{\pi(2)} > \ldots > a_{\pi(k-2)+2} - a_{\pi(k-2)}.$$

  form a monochromatic clique.